Wells

THE

ESSENTIALS OF GEOMETRY

(PLANE)

BY

WEBSTER WELLS, S.B.

PROFESSOR OF MATHEMATICS IN THE MASSACHUSETTS
INSTITUTE OF TECHNOLOGY

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PREFACE.

In the *Essentials of Geometry*, the author has endeavored to prepare a work suited to the needs of high schools and academies. It will also be found to answer as well the requirements of colleges and scientific schools.

In some of its features, the work is similar to the author's *Revised Plane and Solid Geometry;* but important improvements have been introduced, which are in line with the present requirements of many progressive teachers.

In a number of propositions, the figure is given, and a statement of what is to be proved; the details of the proof being left to the pupil, usually with a hint as to the method of demonstration to be employed.

The propositions and corollaries left in this way for the pupil to demonstrate will be found in the following sections: —

Book I., §§ 51, 75, 76, 78, 79, 96, 102, 110, 111, 112, 115, 117, 136.

Book II., §§ 158, 160, 165, 170, 172 (Case III.), 174, 178, 179, 193 (Case III.), 194, and 201.

Book III., §§ 251, 257, 261, 264, 268, 278, 282, 284, and 286.

Book IV., §§ 312 and 316.

Book V., §§ 346, 347, and 350.

There are also Problems in Construction in which the construction or proof is left to the pupil.

Another important improvement consists in giving figures and suggestions for the exercises. In Book I., the pupil has a figure for every non-numerical exercise; after that, they are only given with the more difficult ones.

In many of the exercises in construction, the pupil is expected to discuss the problem, or point out its limitations.

In Book I., the authority for each statement of a proof is given directly after the statement, in smaller type, enclosed in brackets. In the remaining portions of the work, the formal statement of the authority is omitted; but the number of the section where it is to be found is usually given.

In a number of cases, however, where the pupil is presumed, from practice, to be so familiar with the authority as not to require reference to the section where it is to be found, there is given merely an interrogation-point.

In all these cases the pupil should be required to give the authority as carefully and accurately as if it were actually printed on the page.

Another improvement consists in marking the parts of a demonstration by the words *Given, To Prove,* and *Proof,* printed in heavy-faced type. A similar system is followed in the Constructions, by the use of the words *Given, Required, Construction,* and *Proof.*

A minor improvement is the omission of the definite article in speaking of geometrical magnitudes; thus we speak of "angle A," "triangle ABC," etc., and not "the angle A," "the triangle ABC," etc.

Symbols and abbreviations have been freely used; a list of these will be found on page 4.

Particular attention has been given to putting the propositions in the first part of Book I. in a form adapted to the needs of a beginner.

The pages have been arranged in such a way as to avoid the necessity, while reading a proof, of turning the page for reference to the figure.

The Appendix to the Plane Geometry contains propositions on Maxima and Minima of Plane Figures, and Symmetrical Figures; also, additional exercises of somewhat greater difficulty than those previously given.

The author wishes to acknowledge, with thanks, the many suggestions which he has received from teachers in all parts of the country, which have added materially to the value of the work.

WEBSTER WELLS.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
1898.

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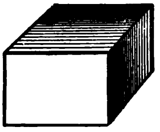
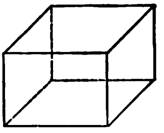
GEOMETRY.

PRELIMINARY DEFINITIONS.



A material body.



A geometrical solid.

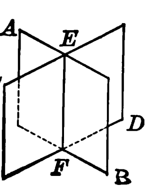
1. A *material body,* such as a block of wood, occupies a *limited* or *bounded* portion of space.

The boundary which separates such a body from surrounding space is called the *surface* of the body.

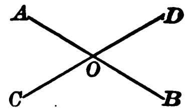
2. If the material composing such a body could be conceived as taken away from it, *without altering the form or shape of the bounding surface,* there would remain a *portion of space* having the same bounding surface as the former material body; this portion of space is called a *geometrical solid,* or simply a *solid.*

The surface which bounds it is called a *geometrical surface,* or simply a *surface;* it is also called the *surface of the solid.*

3. If two geometrical surfaces intersect each other, that which is common to both is called a *geometrical line,* or simply a *line.*



Thus, if surfaces AB and CD cut each other, their common intersection, EF, is a line.



4. If two geometrical lines intersect each other, that which is common to both is called a *geometrical point*, or simply a *point*.

Thus, if lines AB and CD cut each other, their common intersection, O, is a point.

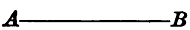
5. A solid has *extension in every direction;* but this is not true of surfaces and lines.

A point has extension in *no* direction, but simply *position in space.*

6. A surface may be conceived as existing independently in space, without reference to the solid whose boundary it forms.

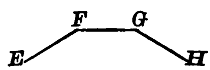
In like manner, we may conceive of lines and points as having an independent existence in space.

7. A *straight line*, or *right line*, is a line which has the same direction throughout its length; as AB.



A straight line.

A curve.



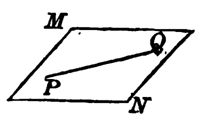
A broken line.

A *curved line*, or *curve*, is a line no portion of which is straight; as CD.

A *broken line* is a line which is composed of different successive straight lines; as $EFGH$.

8. The word "*line*" will be used hereafter as signifying a *straight line.*

9. A *plane surface*, or *plane*, is a surface such that the straight line joining any two of its points lies entirely in the surface.



Thus, if P and Q are any two points in surface MN, and the straight line joining P and Q lies entirely in the surface, then MN is a plane.

10. A *curved surface* is a surface no portion of which is plane.

11. We may conceive of a straight line as being of unlimited extent in regard to length; and in like manner we may conceive of a plane as being of unlimited extent in regard to length and breadth.

12. A *geometrical figure* is any combination of points, lines, surfaces, or solids.

A *plane figure* is a figure formed by points and lines all lying in the same plane.

A geometrical figure is called *rectilinear*, or *right-lined*, when it is composed of straight lines only.

13. *Geometry* treats of the properties, construction, and measurement of geometrical figures.

14. *Plane Geometry* treats of plane figures only.

Solid Geometry, also called *Geometry of Space*, or *Geometry of Three Dimensions*, treats of figures which are not plane.

15. An *Axiom* is a truth which is assumed without proof as being self-evident.

A *Theorem* is a truth which requires demonstration.

A *Problem* is a question proposed for solution.

A *Proposition* is a general term for a theorem or problem.

A *Postulate* assumes the possibility of solving a certain problem.

A *Corollary* is a secondary theorem, which is an immediate consequence of the proposition which it follows.

A *Scholium* is a remark or note.

An *Hypothesis* is a supposition made either in the statement or the demonstration of a proposition.

16. Postulates.

1. We assume that a straight line can be drawn between any two points.
2. We assume that a straight line can be produced (*i.e.*, prolonged) indefinitely in either direction.

17. Axioms.

We assume the truth of the following:

1. *Things which are equal to the same thing, or to equals, are equal to each other.*

2. *If the same operation be performed upon equals, the results will be equal.*

3. *But one straight line can be drawn between two points.*

4. *A straight line is the shortest line between two points.*

5. *The whole is equal to the sum of all its parts.*

6. *The whole is greater than any of its parts.*

18. Since but one straight line can be drawn between two points, a straight line is said to be *determined* by any two of its points.

19. Symbols and Abbreviations.

The following symbols will be used in the work:

+, plus.	△, triangle.
−, minus.	△s, triangles.
×, multiplied by.	⊥, perpendicular, is perpendicular to.
=, equals.	⊥s, perpendiculars.
≎, equivalent, is equivalent to.	∥, parallel, is parallel to.
>, is greater than.	∥s, parallels.
<, is less than.	▱, parallelogram.
∴, therefore.	▱s, parallelograms.
∠, angle.	⊙, circle.
∠s, angles.	Ⓢ, circles.

The following abbreviations will also be used:

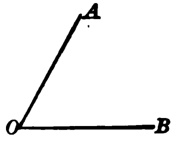
Ax.,	Axiom.	Sup.,	Supplementary.
Def.,	Definition.	Alt.,	Alternate.
Hyp.,	Hypothesis.	Int.,	Interior.
Cons.,	Construction.	Ext.,	Exterior.
Rt.,	Right.	Corresp.,	Corresponding.
Str.,	Straight.	Rect.,	Rectangle, rectangular.
Adj.,	Adjacent.		

PLANE GEOMETRY.

Book I.

RECTILINEAR FIGURES.

DEFINITIONS AND GENERAL PRINCIPLES.



20. An *angle* ($\angle$) is the *amount of divergence* of two straight lines which are drawn from the same point in different directions.

The point is called the *vertex* of the angle, and the straight lines are called its *sides*.

21. If there is but one angle at a given vertex, it may be designated by the letter at that vertex; but if two or more angles have the same vertex, we avoid ambiguity by naming also a letter on each side, placing the letter at the vertex between the others.

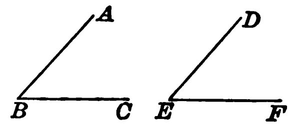
Thus, we should call the angle of § 20 "angle O"; but if there were other angles having the same vertex, we should read it either AOB or BOA.

Another way of designating an angle is by means of a letter placed between its sides; examples of this will be found in § 71.

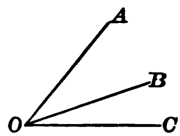
22. Two geometrical figures are said to be *equal* when one can be applied to the other so that they shall coincide throughout.

To prove two *angles* equal, we do not consider the lengths of their sides.

Thus, if angle ABC can be applied to angle DEF in such a manner that point B shall fall on point E, and sides AB and BC on sides DE and EF, respectively, the angles are equal, even if sides AB and BC are not equal in length to sides DE and EF, respectively.



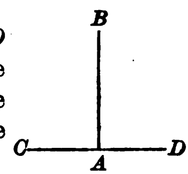
23. Two angles are said to be *adjacent* when they have the same vertex, and a common side between them; as AOB and BOC.



PERPENDICULAR LINES.

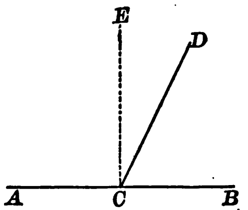
24. If from a given point in a straight line a line be drawn meeting the given line in such a way as to make the adjacent angles equal, each of the equal angles is called a *right angle*, and the lines are said to be *perpendicular* ($\perp$) to each other.

Thus, if from point A in straight line CD line AB be drawn in such a way as to make angles BAC and BAD equal, each of these angles is a right angle, and AB and CD are perpendicular to each other.



PROP. I. THEOREM.

25. *At a given point in a straight line, a perpendicular to the line can be drawn, and but one.*



Let C be the given point in straight line AB.

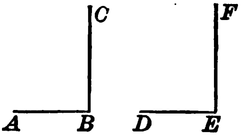
To prove that a perpendicular can be drawn to AB at C, and but one.

Draw a straight line CD in such a position that angle BCD shall be less than angle ACD; and let line CD be turned about point C as a pivot towards the position CA.

Then, angle BCD will constantly increase; and angle ACD will constantly diminish, until it becomes less than angle BCD; and it is evident that there is one position of CD, and only one, in which these angles are equal.

Let CE be this position; then by the definition of § 24, CE is perpendicular to AB.

Hence, a perpendicular can be drawn to AB at C, and but one.



26. Cor. *All right angles are equal.*

Let ABC and DEF be right angles.

To prove angles ABC and DEF equal.

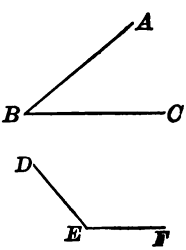
Let angle ABC be superposed (*i.e.*, placed) upon angle DEF in such a way that point B shall fall upon point E, and line AB upon line DE.

Then, line BC will fall upon line EF; for otherwise we should have two lines perpendicular to DE at E, which is impossible.

[At a given point in a straight line, but one perpendicular to the line can be drawn.] (§ 25)

Hence, angles ABC and DEF are equal (§ 22).

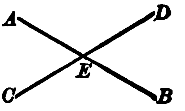
DEFINITIONS.



27. An *acute* angle is an angle which is less than a right angle; as ABC.

An *obtuse* angle is an angle which is greater than a right angle; as DEF.

Acute and obtuse angles are called *oblique* angles; and intersecting lines which are not perpendicular, are said to be oblique to each other.



28. Two angles are said to be *vertical*, or *opposite*, when the sides of one are the prolongations of the sides of the other; as *AEC* and *BED*.

29. An angle is *measured* by finding how many times it contains another angle, adopted arbitrarily as the unit of measure.

The usual unit of measure is the *degree*, which is the ninetieth part of a right angle.

To express fractional parts of the unit, the degree is divided into sixty equal parts called *minutes*, and the minute into sixty equal parts, called *seconds*.

Degrees, minutes, and seconds are represented by the symbols, °, ′, ″, respectively.

Thus, $43° \; 22' \; 37''$ represents an angle of 43 degrees, 22 minutes, and 37 seconds.

30. If the sum of two angles is a right angle, or 90°, one is called the *complement* of the other; and if their sum is two right angles, or 180°, one is called the *supplement* of the other.

For example, the complement of an angle of 34° is $90° - 34°$, or 56°; and the supplement of an angle of 34° is $180° - 34°$, or 146°.

Two angles which are complements of each other are called *complementary;* and two angles which are supplements of each other are called *supplementary*.

31. It is evident that

1. *The complements of equal angles are equal.*
2. *The supplements of equal angles are equal.*

EXERCISES.

1. How many degrees are there in the complement of 47°? of 83°? of 90°?

2. How many degrees are there in the supplement of 31°? of 90°? of 178°?

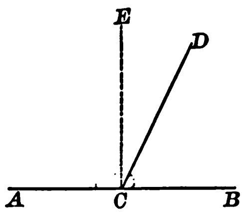
3. How many degrees are there in the complement, and in the supplement, of an angle equal to $\frac{7}{12}$ of a right angle?

4. How many degrees are there in an angle whose supplement is equal to $\frac{4}{3}\frac{1}{}$ of its complement?

5. Two angles are complementary, and the greater exceeds the less by 37°. How many degrees are there in each angle?

PROP. II. THEOREM.

32. *If two adjacent angles have their exterior sides in the same straight line, their sum is equal to two right angles.*



Let angles ACD and BCD have their sides AC and BC in the same straight line.

To prove the sum of angles ACD and BCD equal to two right angles.

Draw line CE perpendicular to AB at C.

[At a given point in a straight line, a perpendicular to the line can be drawn.] (§ 25)

Then, it is evident that the sum of angles ACD and BCD is equal to the sum of angles ACE and BCE.

But since CE is perpendicular to AB, angles ACE and BCE are right angles.

Hence, the sum of angles ACD and BCD is equal to two right angles.

33. **Sch.** Since angles ACD and BCD are *supplementary* (§ 30), the theorem may be stated as follows:

If two adjacent angles have their exterior sides in the same straight line, they are supplementary.

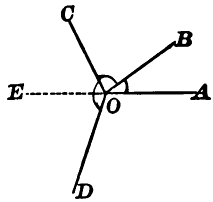
Such angles are called *supplementary-adjacent.*

34. Cor. I. *The sum of all the angles on the same side of a straight line at a given point is equal to two right angles.*

This is evident from § 32.

35. Cor. II. *The sum of all the angles about a point in a plane is equal to four right angles.*

Let *AOB, BOC, COD,* and *DOA* be angles about the point *O*.



To prove the sum of angles *AOB, BOC, COD,* and *DOA* equal to four right angles.

Produce *AO* to *E*.

Then, the sum of angles *AOB, BOC,* and *COE* is equal to two right angles.

[The sum of all the angles on the same side of a straight line at a given point is equal to two right angles.] (§ 34)

In like manner, the sum of angles *EOD* and *DOA* is equal to two right angles.

Therefore, the sum of angles *AOB, BOC, COD,* and *DOA* is equal to four right angles.

Ex. 6. If, in the figure of § 35, angles *AOB*, *BOC*, and *COD* are respectively 49°, 88°, and $\frac{8}{9}$ of a right angle, how many degrees are there in angle *AOD*?

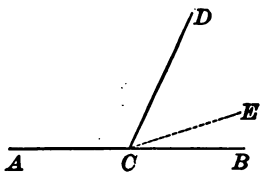
36. Sch. The pupil will now observe that a demonstration, in Geometry, consists of three parts:

1. *The statement of what is given in the figure.*
2. *The statement of what is to be proved.*
3. *The proof.*

In the remaining propositions of the work, we shall mark clearly the three divisions of the demonstration by heavy-faced type, and employ the symbols and abbreviations of § 20.

PROP. III. THEOREM.

37. *If the sum of two adjacent angles is equal to two right angles, their exterior sides lie in the same straight line.*



Given the sum of adj. $\angle s$ ACD and BCD equal to two rt. $\angle s$.

To Prove that AC and BC lie in the same str. line.

Proof. If AC and BC do not lie in the same str. line, let CE be in the same str. line with AC.

Then since ACE is a str. line, $\angle ECD$ is the supplement of $\angle ACD$.

[If two adj. $\angle s$ have their ext. sides in the same str. line, they are supplementary.] (§ 33)

But by hyp., $\angle ACD + \angle BCD =$ two rt. $\angle s$.

Whence, $\angle BCD$ is the supplement of $\angle ACD$. (§ 30)

Then since both $\angle ECD$ and $\angle BCD$ are supplements of $\angle ACD$, $\angle ECD = \angle BCD$.

[The supplements of equal $\angle s$ are equal.] (§ 31)

Hence, EC coincides with BC, and AC and BC lie in the same str. line.

38. Sch. I. It will be observed that the enunciation of every theorem consists essentially of two parts; the *Hypothesis*, and the *Conclusion*.

Thus, we may enunciate Prop. I as follows:

Hypothesis. If a point be taken in a given straight line,

Conclusion. A perpendicular to the line at the given point can be drawn, and but one.

39. Sch. II. We may enunciate Prop. II as follows:

Hypothesis. If two adjacent angles have their exterior sides in the same straight line,

Conclusion. Their sum is equal to two right angles.

Again, we may enunciate Prop. III:

Hypothesis. If the sum of two adjacent angles is equal to two right angles,

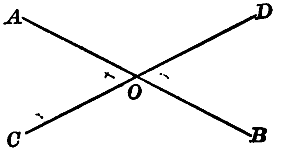
Conclusion. Their exterior sides lie in the same straight line.

One proposition is said to be the **Converse** of another when the hypothesis and conclusion of the first are, respectively, the conclusion and hypothesis of the second.

It is evident from the above considerations that Prop. III is the *converse* of Prop. II.

Prop. IV. Theorem.

40. *If two straight lines intersect, the vertical angles are equal.*



Given str. lines AB and CD intersecting at O.

To Prove $\angle AOC = \angle BOD$.

Proof. Since $\angle$s AOC and AOD have their ext. sides in str. line CD, $\angle AOC$ is the supplement of $\angle AOD$.

[If two adj. $\angle$s have their ext. sides in the same str. line, they are supplementary.] (§ 33)

For the same reason, $\angle BOD$ is the supplement of $\angle AOD$.

$$\therefore \angle AOC = \angle BOD.$$

[The supplements of equal $\angle$s are equal.] (§ 31)

In like manner, we may prove

$$\angle AOD = \angle BOC.$$

EXERCISES.

7. If, in the figure of Prop. IV., $\angle AOD = 137°$, how many degrees are there in BOC? in AOC? in BOD?

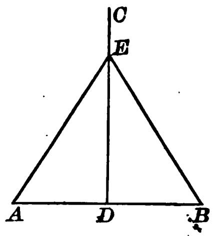
8. Two angles are supplementary, and the greater is seven times the less. How many degrees are there in each angle?

PROP. V. THEOREM.

41. *If a perpendicular be erected at the middle point of a straight line,*

I. *Any point in the perpendicular is equally distant from the extremities of the line.*

II. *Any point without the perpendicular is unequally distant from the extremities of the line.*



I. **Given** line $CD \perp$ to line AB at its middle point D, E any point in CD, and lines AE and BE.

To Prove $AE = BE$.

Proof. Superpose figure BDE upon figure ADE by folding it over about line DE as an axis.

Now $\angle BDE = \angle ADE$.

[All rt. $\angle$s are equal.] (§ 26)

Then, line BD will fall upon line AD.

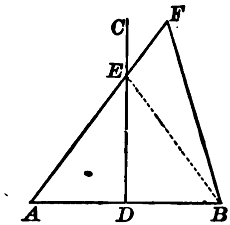
But by hyp., $BD = AD$.

Whence, point B will fall on point A.

Then line BE will coincide with line AE.

[But one str. line can be drawn between two points.] (Ax. 3)

$\therefore AE = BE$.



II. **Given** line $CD \perp$ to line AB at its middle point D, F any point without CD, and lines AF and BF.

To Prove $AF > BF$.

Proof. Let AF intersect CD at E, and draw line BE.

Now $BE + EF > BF$.

[A str. line is the shortest line between two points.] (Ax. 4)

But, $BE = AE$.

[If a $\perp$ be erected at the middle point of a str. line, any point in the $\perp$ is equally distant from the extremities of the line.] (§ 41, I)

Substituting for BE its equal AE, we have

$$AE + EF > BF, \text{ or } AF > BF.$$

42. Cor. I. *Every point which is equally distant from the extremities of a straight line, lies in the perpendicular erected at the middle point of the line.*

43. Cor. II. Since a straight line is determined by any two of its points (§ 18), it follows from § 42 that

Two points, each equally distant from the extremities of a straight line, determine a perpendicular at its middle point.

44. Cor. III. When figure BDE is superposed upon figure ADE, in the proof of § 41, I., $\angle EBD$ coincides with $\angle EAD$, and $\angle BED$ with $\angle AED$.

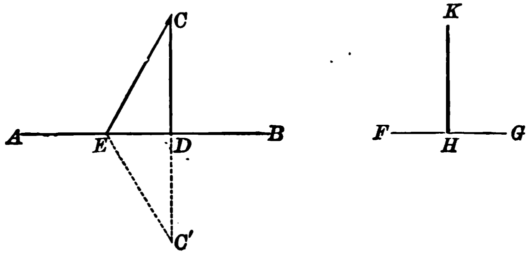
That is, $\angle EAD = \angle EBD$, and $\angle AED = \angle BED$.

Then, *if lines be drawn to the extremities of a straight line from any point in the perpendicular erected at its middle point,*

1. *They make equal angles with the line.*
2. *They make equal angles with the perpendicular.*

PROP. VI. THEOREM.

45. *From a given point without a straight line, a perpendicular can be drawn to the line, and but one.*



Given point C without line AB.

To Prove that a $\perp$ can be drawn from C to AB, and but one.

Proof. Let line HK be $\perp$ to line FG at H.

[At a given point in a str. line, a $\perp$ to the line can be drawn.] (§ 25)

Apply line FG to line AB, and move it along until HK passes through C; let point H fall at D, and draw line CD.

Then, CD is $\perp$ AB.

If possible, let CE be another $\perp$ from C to AB.

Produce CD to C', making $C'D = CD$, and draw line EC'.

By cons., ED is $\perp$ to CC' at its middle point D.

$$\therefore \angle CED = \angle C'ED.$$

[If lines be drawn to the extremities of a str. line from any point in the $\perp$ erected at its middle point, they make equal $\angle$s with the $\perp$.] (§ 44)

But by hyp., $\angle CED$ is a rt. $\angle$; then, $\angle C'ED$ is a rt. $\angle$.

$$\therefore \angle CED + \angle C'ED = \text{two rt. } \angle s.$$

Then line CEC' is a str. line.

[If the sum of two adj. $\angle$s is equal to two rt. $\angle$s, their ext. sides lie in the same str. line.] (§ 37)

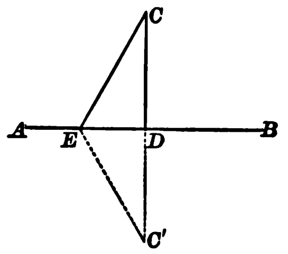
But this is impossible, for, by cons., CDC' is a str. line.

[But one str. line can be drawn between two points.] (Ax. 3)

Hence, CE cannot be $\perp$ AB, and CD is the only $\perp$ that can be drawn.

PROP. VII. THEOREM.

46. *The perpendicular is the shortest line that can be drawn from a point to a straight line.*



Given CD the $\perp$ from point C to line AB, and CE any other str. line from C to AB.

To Prove $CD < CE$.

Proof. Produce CD to C', making $C'D = CD$, and draw line EC'.

By cons., ED is $\perp$ to CC' at its middle point D.

$$\therefore CE = C'E.$$

[If a $\perp$ be erected at the middle point of a str. line, any point in the $\perp$ is equally distant from the extremities of the line.] (§ 41)

But $CD + DC' < CE + EC'$.

[A str. line is the shortest line between two points.] (Ax 4.)

Substituting for DC' and EC' their equals CD and CE, respectively, we have

$$2\,CD < 2\,CE.$$

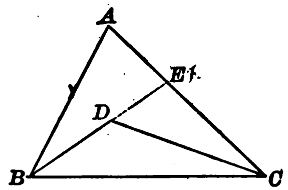
$$\therefore CD < CE.$$

47. Sch. The *distance* of a point from a line is understood to mean the length of the perpendicular from the point to the line.

Ex. 9. Find the number of degrees in the angle the sum of whose supplement and complement is 196°.

Prop. VIII. Theorem.

48. *If two lines be drawn from a point to the extremities of a straight line, their sum is greater than the sum of two other lines similarly drawn, but enveloped by them.*



Given lines AB and AC drawn from point A to the extremities of line BC; and DB and DC two other lines similarly drawn, but enveloped by AB and AC.

To Prove $AB + AC > DB + DC.$

Proof. Produce BD to meet AC at E.

Now $AB + AE > BE.$

[A str. line is the shortest line between two points.] **(Ax. 4)**

Adding EC to both members of the inequality,

$$BA + AC > BE + EC.$$

Again, $DE + EC > DC.$

Adding BD to both members of the inequality,

$$BE + EC > BD + DC.$$

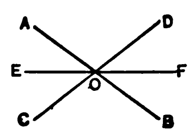
Since $BA + AC$ is greater than $BE + EC$, which is itself greater than $BD + DC$, it follows that

$$AB + AC > DB + DC.$$

EXERCISES.

10. The straight line which bisects an angle bisects also its vertical angle.

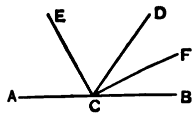
(If OE bisects $\angle AOC$, $\angle AOE = \angle COE$; and these $\angle$s are equal to $\angle$s BOF and DOF, respectively.)



11. The bisectors of a pair of vertical angles lie in the same straight line.

(Fig. of Ex. 10. To prove EOF a str. line. $\angle COE = \angle DOF$, for they are the halves of equal $\angle$s; but $\angle DOE + \angle COE = 2$ rt. $\angle$s, and therefore $\angle DOE + \angle DOF = 2$ rt. $\angle$s.)

12. The bisectors of two supplementary adjacent angles are perpendicular to each other.

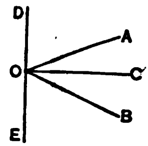


(We have $\angle ACD + \angle BCD = 2$ rt. $\angle$s; and $\angle$s DCE and DCF are the halves of $\angle$s ACD and BCD, respectively.)

13. If the bisectors of two adjacent angles are perpendicular, the angles are supplementary.

(Fig. of Ex. 12. Sum of $\angle$s DCE and $DCF = 1$ rt. $\angle$, and $\angle$s DCE and DCF are the halves of $\angle$s ACD and BCD, respectively.)

14. A line drawn through the vertex of an angle perpendicular to its bisector makes equal angles with the sides of the given angle.



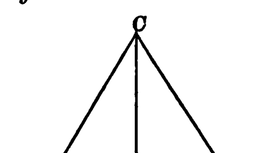
($\angle$s AOD and BOE are complements of $\angle$s AOC and BOC, respectively.)

PROP. IX. THEOREM.

49. *If oblique lines be drawn from a point to a straight line,*

I. *Two oblique lines cutting off equal distances from the foot of the perpendicular from the point to the line are equal.*

II. *Of two oblique lines cutting off unequal distances from the foot of the perpendicular from the point to the line, the more remote is the greater.*



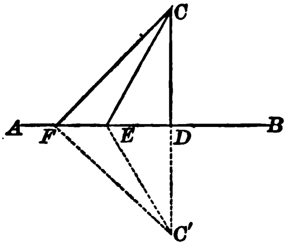
I. **Given** CD the $\perp$ from point C to line AB; and CE and CF oblique lines from C to AB, cutting off equal distances from the foot of CD.

To Prove $CE = CF.$

Proof. By hyp., CD is $\perp$ to EF at its middle point D.

$$\therefore CE = CF.$$

[If a $\perp$ be erected at the middle point of a str. line, any point in the $\perp$ is equally distant from the extremities of the line.] (§ 41)



II. **Given** CD the $\perp$ from point C to line AB; and CE and CF oblique lines from C to AB, cutting off unequal distances from the foot of CD; CF being the more remote.

To Prove $CF > CE.$

Proof. Produce CD to C', making $C'D = CD$, and draw lines $C'E$ and $C'F$.

By cons., AD is $\perp$ to CC' at its middle point D.

$$\therefore CF = C'F, \text{ and } CE = C'E.$$

[If a $\perp$ be erected at the middle point of a str. line, any point in the $\perp$ is equally distant from the extremities of the line.] (§ 41)

But $CF + FC' > CE + EC'.$

[If two lines be drawn from a point to the extremities of a str. line, their sum is $>$ the sum of two other lines similarly drawn, but enveloped by them.] (§ 48)

Substituting for FC' and EC' their equals CF and CE, respectively, we have

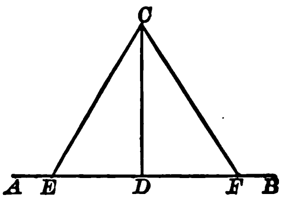
$$2\,CF > 2\,CE.$$

$$\therefore CF > CE.$$

Note. The theorem holds equally if oblique line CE is on the opposite side of perpendicular CD from CF.

PROP. X. THEOREM.

50. (Converse of Prop. IX., I.) *If oblique lines be drawn from a point to a straight line, two equal oblique lines cut off equal distances from the foot of the perpendicular from the point to the line.*



Given CD the $\perp$ from point C to line AB, and CE and CF equal oblique lines from C to AB.

To Prove $DE = DF$.

Proof. We know that DE is either $>$, equal to, or $<$ DF.

If we suppose $DE > DF$, CE would be $> CF$.

[If oblique lines be drawn from a point to a str. line, of two oblique lines cutting off unequal distances from the foot of the $\perp$ from the point to the line, the more remote is the greater.] (§ 49)

But this is contrary to the hypothesis that $CE = CF$.

Hence, DE cannot be $> DF$.

In like manner, if we suppose $DE < DF$, CE would be $< CF$, which is contrary to the hypothesis that $CE = CF$.

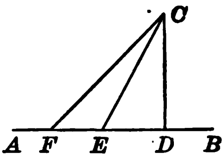
Hence, DE cannot be $< DF$.

Then, if DE can be neither $> DF$, nor $< DF$, we must have $DE = DF$.

Note. The method of proof exemplified in Prop. X is known as the "Indirect Method," or the "*Reductio ad Absurdum.*"

The truth of a proposition is demonstrated by making every possible supposition in regard to the matter, and showing that, in all cases except the one which we wish to prove, the supposition leads to something which is contrary to the hypothesis.

51. Cor. (Converse of Prop. IX, II.) *If two unequal oblique lines be drawn from a point to a straight line, the greater cuts off the greater distance from the foot of the perpendicular from the point to the line.*



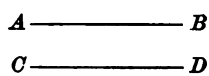
Given CD the $\perp$ from point C to line AB; and CE and CF unequal oblique lines from C to AB, CF being $> CE$.

To Prove $DF > DE$.

(Prove by *Reductio ad Absurdum;* by § 49, I, DE cannot equal DF, and by § 49, II, it cannot be $> DF$.)

PARALLEL LINES.

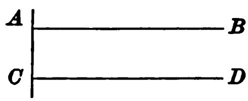
52. Def. Two straight lines are said to be *parallel* (‖) when they lie in the same plane, and cannot meet however far they may be produced; as AB and CD.



53. Ax. *We assume that but one straight line can be drawn through a given point parallel to a given straight line.*

PROP. XI. THEOREM.

54. *Two perpendiculars to the same straight line are parallel.*



Given lines AB and $CD \perp$ to line AC.

To Prove $AB \parallel CD$.

Proof. If AB and CD are not ‖, they will meet in some point if sufficiently produced (§ 52).

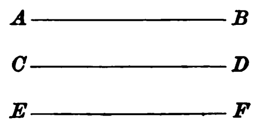
We should then have two ⊥s from this point to AC, which is impossible.

[From a given point without a str. line, but one $\perp$ can be drawn to the line.] (§ 45)

Therefore, AB and CD cannot meet, and are ‖.

PROP. XII. THEOREM.

55. *Two straight lines parallel to the same straight line are parallel to each other.*



Given lines AB and $CD \parallel$ to line EF.

To Prove $AB \parallel CD$.

Proof. If AB and CD are not $\parallel$, they will meet in some point if sufficiently produced. (§ 52)

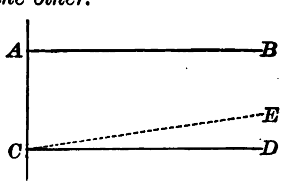
We should then have two lines drawn through this point $\parallel$ to EF, which is impossible.

[But one str. line can be drawn through a given point $\parallel$ to a given str. line.] (§ 53)

Therefore, AB and CD cannot meet, and are $\parallel$.

PROP. XIII. THEOREM.

56. *A straight line perpendicular to one of two parallels is perpendicular to the other.*



Given lines AB and $CD \parallel$, and line $AC \perp AB$.

To Prove $AC \perp CD$.

Proof. If CD is not $\perp AC$, let line CE be $\perp AC$.

Then since AB and CE are $\perp AC$, $CE \parallel AB$.

[Two $\perp$s to the same str. line are $\parallel$.] (§ 54)

But by hyp., $CD \parallel AB$.

Then, CE must coincide with CD.

[But one str. line can be drawn through a given point $\parallel$ to a given str. line.] (§ 53)

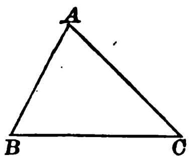
But by cons., $AC \perp CE$.

Then since CE coincides with CD, we have $AC \perp CD$.

TRIANGLES.

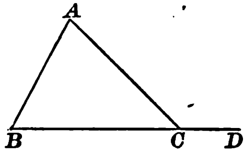
DEFINITIONS.

57. A *triangle* ($\triangle$) is a portion of a plane bounded by three straight lines; as ABC.



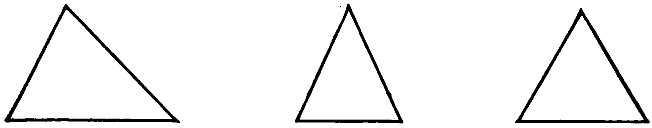
The bounding lines, AB, BC, and CA, are called the *sides* of the triangle, and their points of intersection, A, B, and C, the *vertices*.

The *angles* of the triangle are the angles CAB, ABC, and BCA, included between the adjacent sides.



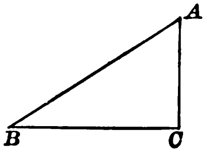
An *exterior angle* of a triangle is the angle at any vertex between any side of the triangle and the adjacent side produced; as ACD.

58. A triangle is called *scalene* when no two of its sides are equal; *isosceles* when two of its sides are equal; *equilateral* when all its sides are equal; and *equiangular* when all its angles are equal.



Scalene. *Isosceles.* *Equilateral.*

59. A *right triangle* is a triangle which has a right angle; as ABC, which has a right angle at C.

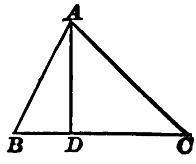


The side AB opposite the right angle is called the *hypotenuse*, and the other sides, AC and BC, the *legs*.

60. If any side of a triangle be taken and called the *base*, the corresponding *altitude* is the perpendicular drawn from the opposite vertex to the base, produced if necessary.

In general, either side may be taken as the base; but in an isosceles triangle, unless otherwise specified, the side which is not one of the equal sides is taken as the base.

When any side has been taken as the base, the opposite angle is called the *vertical angle*, and its vertex is called the *vertex of the triangle*.



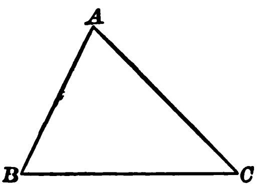
Thus, in triangle ABC, BC is the base, AD the altitude, and BAC the vertical angle.

61. Since a straight line is the shortest line between two points (Ax. 4), it follows that

Any side of a triangle is less than the sum of the other two sides.

PROP. XIV. THEOREM.

62. *Any side of a triangle is greater than the difference of the other two sides.*



Given AB, any side of $\triangle ABC$; and side $BC >$ side AC.

To Prove $AB > BC - AC$.

Proof. We have $AB + AC > BC$.

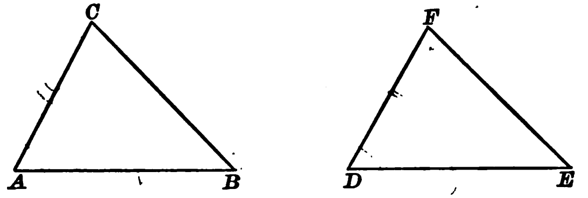
[A str. line is the shortest line between two points.] (Ax. 4)

Subtracting AC from both members of the inequality,

$$AB > BC - AC.$$

PROP. XV. THEOREM.

63. *Two triangles are equal when two sides and the included angle of one are equal respectively to two sides and the included angle of the other.*



Given, in $\triangle ABC$ and DEF,

$$AB = DE,\ AC = DF, \text{ and } \angle A = \angle D.$$

To Prove $\triangle ABC = \triangle DEF$.

Proof. Superpose $\triangle ABC$ upon $\triangle DEF$ in such a way that $\angle A$ shall coincide with its equal $\angle D$; side AB falling on side DE, and side AC on side DF.

Then since $AB = DE$ and $AC = DF$, point B will fall on point E, and point C on point F.

Whence, side BC will coincide with side EF.

[But one str. line can be drawn between two points.] (Ax. 3)

Therefore, the $\triangle$ coincide throughout, and are equal.

64. Cor. Since ABC and DEF coincide throughout, we have $\angle B = \angle E$, $\angle C = \angle F$, and $BC = EF$.

65. Sch. I. In equal figures, lines or angles which are similarly placed are called *homologous*.

Thus, in the figure of Prop. XV, $\angle A$ is homologous to $\angle D$; AB is homologous to DE; etc.

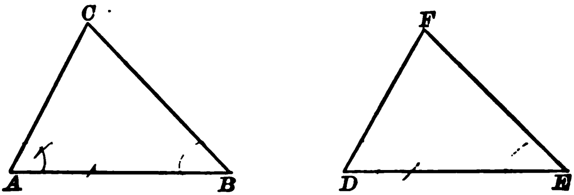
66. Sch. II. It follows from § 65 that

In equal figures, the homologous parts are equal.

67. Sch. III. In equal triangles, the equal angles lie opposite the equal sides.

PROP. XVI. THEOREM.

68. ***Two triangles are equal when a side and two adjacent angles of one are equal respectively to a side and two adjacent angles of the other.***



Given, in $\triangle ABC$ and DEF,

$$AB = DE, \ \angle A = \angle D, \text{ and } \angle B = \angle E.$$

To Prove $\triangle ABC = \triangle DEF$.

Proof. Superpose $\triangle ABC$ upon $\triangle DEF$ in such a way that side AB shall coincide with its equal DE; point A falling on point D, and point B on point E.

Then since $\angle A = \angle D$, side AC will fall on side DF, and point C will fall somewhere on DF.

And since $\angle B = \angle E$, side BC will fall on side EF, and point C will fall somewhere on EF.

Then point C, falling at the same time on DF and EF, must fall at their intersection, F.

Therefore, the $\triangle$ coincide throughout, and are equal.

EXERCISES.

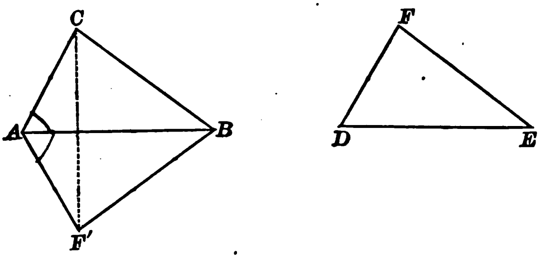
15. If, in the figure of Prop. XV., $AB = EF$, $BC = DE$, and $\angle B = \angle E$, which angle of triangle DEF is equal to A? which angle is equal to C?

16. If, in the figure of Prop. XVI., $AC = DF$, $\angle A = \angle F$, and $\angle C = \angle D$, which side of triangle DEF is equal to AB? which side is equal to BC?

17. If OD and OE are the bisectors of two complementary-adjacent angles, AOB and BOC, how many degrees are there in $\angle DOE$?

Prop. XVII. Theorem.

69. *Two triangles are equal when the three sides of one are equal respectively to the three sides of the other.*



Given, in ▵ ABC and DEF,

$$AB = DE,\ BC = EF, \text{ and } CA = FD.$$

To Prove $\triangle ABC = \triangle DEF$.

Proof. Place $\triangle DEF$ in the position ABF'; side DE coinciding with its equal AB, and vertex F falling at F', on the opposite side of AB from C.

Draw line CF'.

By hyp., $AC = AF'$ and $BC = BF'$.

Whence, AB is $\perp$ to CF' at its middle point.

[Two points, each equally distant from the extremities of a str. line, determine a $\perp$ at its middle point.] (§ 43)

$$\therefore \angle BAC = \angle BAF'.$$

[If lines be drawn to the extremities of a str. line from any point in the $\perp$ erected at its middle point, they make equal ∠s with the $\perp$.] (§ 44)

Then since sides AB and AC and $\angle BAC$ of $\triangle ABC$ are equal, respectively, to sides AB and AF' and $\angle BAF'$ of $\triangle ABF'$,

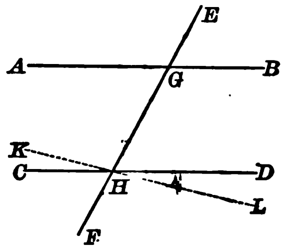
$$\triangle ABC = \triangle ABF'.$$

[Two ▵ are equal when two sides and the included ∠ of one are equal respectively to two sides and the included ∠ of the other.] (§ 63)

That is, $\triangle ABC = \triangle DEF$.

PROP. XX. THEOREM.

73. (Converse of Prop. XIX.) *If two straight lines are cut by a transversal, and the alternate-interior angles are equal, the two lines are parallel.*



Given lines AB and CD cut by transversal EF **at points** G and H, respectively, and

$$\angle AGH = \angle GHD.$$

To Prove $AB \parallel CD.$

Proof. If CD is not $\parallel$ AB, draw line KL through $H \parallel AB$. Then since $\parallel$s AB and KL are cut by transversal EF,

$$\angle AGH = \angle GHL.$$

[If two $\parallel$s are cut by a transversal, the alt. int. $\angle$s are equal.] (§ 72)

But by hyp., $\angle AGH = \angle GHD.$

$$\therefore \angle GHL = \angle GHD.$$

[Things which are equal to the same thing are equal to each other.] (Ax. 1)

But this is impossible unless KL coincides with CD.

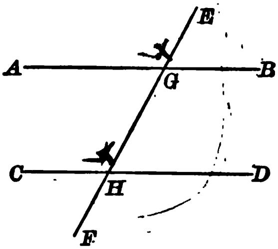
$$\therefore CD \parallel AB.$$

In like manner, it may be proved that if AB and CD are cut by EF, and $\angle BGH = \angle CHG$, then $AB \parallel CD$.

Ex. 18. If, in the figure of Prop. XIX., $\angle AGH = 68°$, how many degrees are there in BGH? in GHD? in DHF?

PROP. XXI. THEOREM.

74. *If two parallels are cut by a transversal, the corresponding angles are equal.*



Given ||s AB and CD cut by transversal EF at points G and H, respectively.

To Prove $\angle AGE = \angle CHG$.

Proof. We have $\angle BGH = \angle CHG$.

[If two ||s are cut by a transversal, the alt. int. ∠s are equal.] (§ 72)

But, $\angle BGH = \angle AGE$.

[If two str. lines intersect, the vertical ∠s are equal.] (§ 40)

$\therefore \angle AGE = \angle CHG$.

[Things which are equal to the same thing are equal to each other.] (Ax. 1)

In like manner, we may prove

$\angle AGH = \angle CHF$, $\angle BGE = \angle DHG$, and $\angle BGH = \angle DHF$.

75. Cor. I. *If two parallels are cut by a transversal, the alternate-exterior angles are equal.*

(Fig. of Prop. XXI.)

Given ||s AB and CD cut by transversal EF at points G and H, respectively.

To Prove $\angle AGE = \angle DHF$.

($\angle BGH = \angle CHG$, and the theorem follows by § 40.)

What other two ext. ∠s in the figure are equal?

76. Cor. II. *If two parallels are cut by a transversal, the sum of the interior angles on the same side of the transversal is equal to two right angles.*

(Fig. of Prop. XXI.)

Given ∥s AB and CD cut by transversal EF at points G and H, respectively.

To Prove $\angle AGH + \angle CHG =$ two rt. ∠s.

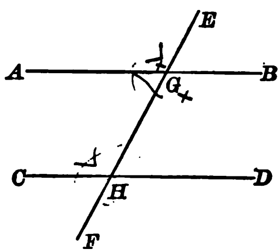
(By § 32, $\angle AGH + \angle AGE \cong$ two rt. ∠s; the theorem follows by § 74.)

What other two int. ∠s in the figure have their sum equal to two rt. ∠s?

X

Prop. XXII. Theorem.

77. (Converse of Prop. XXI.) *If two straight lines are cut by a transversal, and the corresponding angles are equal, the two lines are parallel.*



Given lines AB and CD cut by transversal EF at points G and H, respectively, and

$$\angle AGE = \angle CHG.$$

To Prove $AB \parallel CD$.

Proof. We have $\angle AGE = \angle BGH$.

[If two str. lines intersect, the vertical ∠s are equal.] (§ 40)

$$\therefore \angle BGH = \angle CHG.$$

[Things which are equal to the same thing are equal to each other.] (Ax. 1)

$\therefore\ AB \parallel CD.$

[If two str. lines are cut by a transversal, and the alt. int. ∠s are equal, the two lines are ∥.] (§ 73)

In like manner, it may be proved that if

$\angle AGH = \angle CHF$, or $\angle BGE = \angle DHG$, or $\angle BGH = \angle DHF$,

then $AB \parallel CD.$

78. Cor. I. (Converse of § 75.) *If two straight lines are cut by a transversal, and the alternate-exterior angles are equal, the two lines are parallel.*

(Fig. of Prop. XXII.)

Given lines AB and CD cut by transversal EF at points G and H, respectively, and

$$\angle AGE = \angle DHF.$$

To Prove $AB \parallel CD.$

($\angle AGE = \angle BGH$, and $\angle DHF = \angle CHG$; and the theorem follows by § 73.)

What other two ext. ∠s are there in the figure such that, if they are equal, $AB \parallel CD$?

79. Cor. II. (Converse of § 76.) *If two straight lines are cut by a transversal, and the sum of the interior angles on the same side of the transversal is equal to two right angles, the two lines are parallel.*

(Fig. of Prop. XXII.)

Given lines AB and CD cut by transversal EF at points G and H, respectively, and

$$\angle AGH + \angle CHG = \text{two rt. } \angle s.$$

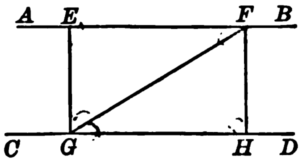
To Prove $AB \parallel CD.$

($\angle CHG$ is the supplement of $\angle AGH$, and also of $\angle GHD$; then ∠s AGH and GHD are equal by § 31, 2, and the theorem follows by § 73.)

What other two int. ∠s are there in the figure such that, if their sum equals two rt. ∠s, $AB \parallel CD$?

PROP. XXIII. THEOREM.

80. *Two parallel lines are everywhere equally distant.*



Given $\parallel$s AB and CD, E and F any two points on AB, and EG and FH lines $\perp$ CD.

To Prove $EG = FH$ (§ 47).

Proof. Draw line FG.

We have $EG \perp AB$.

[A str. line $\perp$ to one of two $\parallel$s is $\perp$ to the other.] (§ 56)

Then, in rt. $\triangle$s EFG and FGH,

$$FG = FG.$$

And since $\parallel$s AB and CD are cut by FG,

$$\angle EFG = \angle FGH.$$

[If two $\parallel$s are cut by a transversal, the alt. int. $\angle$s are equal.] (§ 72)

$$\therefore \triangle EFG = \triangle FGH.$$

[Two rt. $\triangle$s are equal when the hypotenuse and an adj. $\angle$ of one are equal respectively to the hypotenuse and an adj. $\angle$ of the other.] (§ 70)

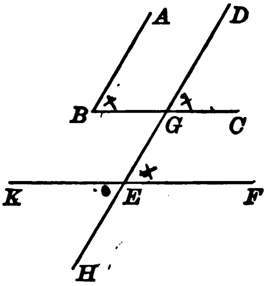
$$\therefore EG = FH.$$

[In equal figures, the homologous parts are equal.] (§ 66)

PROP. XXIV. THEOREM.

81. *Two angles whose sides are parallel, each to each, are equal if both pairs of parallel sides extend in the same direction, or in opposite directions, from their vertices.*

Note. The sides extend in the *same* direction if they are on the *same* side of a straight line joining the vertices, and in *opposite* directions if they are on *opposite* sides of this line.



Given lines AB and BC ‖ to lines DH and KF, respectively, intersecting at E.

I. **To Prove** that ∡ ABC and DEF, whose sides AB and DE, and also BC and EF, extend in the same direction from their vertices, are equal.

Proof. Let BC and DH intersect at G.

Since ‖s AB and DE are cut by BC,

$$\angle ABC = \angle DGC.$$

[If two ‖s are cut by a transversal, the corresp. ∡ are equal.] (§ 74)

In like manner, since ‖s BC and EF are cut by DE,

$$\angle DGC = \angle DEF.$$

$$\therefore \angle ABC = \angle DEF. \quad (1)$$

[Things which are equal to the same thing are equal to each other.] (Ax. 1)

II. **To Prove** that ∡ ABC and HEK, whose sides AB and EH, and also BC and EK, extend in opposite directions from their vertices, are equal.

Proof. From (1), $\angle ABC = \angle DEF.$

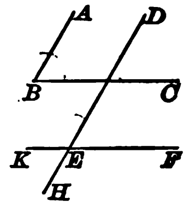
But, $\angle DEF = \angle HEK.$

[If two str. lines intersect, the vertical ∡ are equal.] (§ 40)

$$\therefore \angle ABC = \angle HEK.$$

[Things which are equal to the same thing, are equal to each other.] (Ax. 1)

82. Cor. *Two angles whose sides are parallel, each to each, are supplementary if one pair of parallel sides extend in the same direction, and the other pair in opposite directions, from their vertices.*



Given lines AB and BC $\parallel$ to lines DH and KF, respectively, intersecting at E.

To Prove that $\angle$s ABC and DEK, whose sides AB and DE extend in the same direction, and BC and EK in opposite directions, from their vertices, are supplementary.

Proof. We have $\angle ABC = \angle DEF$.

[Two $\angle$s whose sides are $\parallel$, each to each, are equal if both pairs of $\parallel$ sides extend in the same direction from their vertices.] (§ 81)

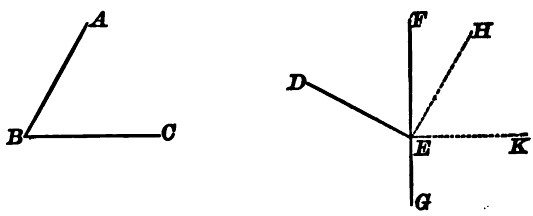
But $\angle DEF$ is the supplement of $\angle DEK$.

[If two adj. $\angle$s have their ext. sides in the same str. line, they are supplementary.] (§ 33)

Then its equal, $\angle ABC$, is the supplement of $\angle DEK$.

PROP. XXV. THEOREM.

83. *Two angles whose sides are perpendicular, each to each, are either equal or supplementary.*



Given lines AB and $BC \perp$ to lines DE and FG, respectively, intersecting at E.

To Prove $\angle ABC$ equal to $\angle DEF$, and supplementary to $\angle DEG$.

Proof. Draw line $EH \perp DE$, and line $EK \perp EF$.

Then since EH and AB are $\perp$ DE,

$$EH \parallel AB.$$

[Two $\perp$s to the same str. line are $\parallel$.] (§ 54)

In like manner, since EK and BC are $\perp$ EF,

$$EK \parallel BC.$$

$$\therefore \angle HEK = \angle ABC.$$

[Two ∠s whose sides are $\parallel$, each to each, are equal if both pairs of $\parallel$ sides extend in the same direction from their vertices.] (§ 81)

But since, by cons., ∠s DEH and FEK are rt. ∠s, each of the ∠s DEF and HEK is the complement of $\angle FEH$.

$$\therefore \angle DEF = \angle HEK.$$

[The complements of equal ∠s are equal.] (§ 31)

$$\therefore \angle ABC = \angle DEF.$$

[Things which are equal to the same thing are equal to each other.] (Ax. 1)

Again, $\angle DEF$ is the supplement of $\angle DEG$.

[If two adj. ∠s have their ext. sides in the same str. line, they are supplementary.] (§ 33)

Then, its equal, $\angle ABC$ is the supplement of $\angle DEG$.

Note. The angles are equal if they are both acute or both obtuse; and supplementary if one is acute and the other obtuse.

EXERCISES.

19. If, in the figure of Prop. XXIV., $\angle ABC = 59°$, how many degrees are there in each of the angles formed about the point E?

20. The line passing through the vertex of an angle perpendicular to its bisector bisects the supplementary adjacent angle.

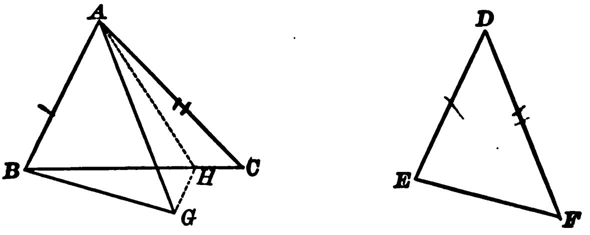
(Fig. of Ex. 12. Let CE bisect $\angle ACD$, and suppose $CF \perp CE$; sum of ∠s ACD and $BCD = 2$ rt. ∠s; then sum of ∠s DCE and $\frac{1}{2} BCD = 1$ rt. ∠; but sum of ∠s DCE and DCF is also 1 rt. ∠; whence the theorem follows.)

21. Any side of a triangle is less than the half-sum of the sides of the triangle.

(Fig. of Prop. XIV. We have $AB < BC + CA$; then add AB to both members of the inequality.)

Prop. XXVIII. Theorem.

91. *If two triangles have two sides of one equal respectively to two sides of the other, but the included angle of the first greater than the included angle of the second, the third side of the first is greater than the third side of the second.*



Given, in ▲ ABC and DEF,

$$AB = DE,\ AC = DF, \text{ and } \angle BAC > \angle D.$$

To Prove $BC > EF$.

Proof. Place $\triangle DEF$ in the position ABG; side DE coinciding with its equal AB, and vertex F falling at G.

Draw line AH bisecting $\angle GAC$, and meeting BC at H; also, draw line GH.

In ▲ AGH and ACH, $AH = AH$.

Also, by hyp., $AG = AC$.

And by cons., $\angle GAH = \angle CAH$.

$$\therefore \triangle AGH = \triangle ACH.$$

[Two ▲ are equal when two sides and the included $\angle$ of one are equal respectively to two sides and the included $\angle$ of the other.] (§ 63)

$$\therefore GH = CH.$$

[In equal figures, the homologous parts are equal.] (§ 66)

But, $BH + GH > BG$.

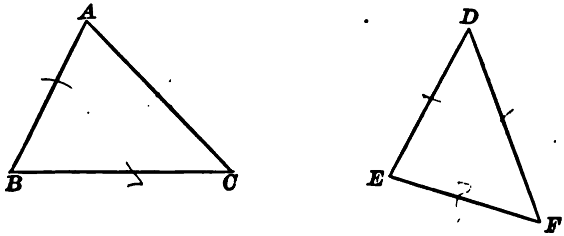
[A str. line is the shortest line between two points.] (Ax. 4)

Substituting for GH its equal CH, we have

$$BH + CH > BG, \text{ or } BC > EF.$$

PROP. XXIX. THEOREM.

92. (Converse of Prop. XXVIII.) *If two triangles have two sides of one equal respectively to two sides of the other, but the third side of the first greater than the third side of the second, the included angle of the first is greater than the included angle of the second.*



Given, in ⊿s ABC and DEF,

$$AB = DE,\ AC = DF, \text{ and } BC > EF.$$

To Prove $\angle A > \angle D$.

Proof. We know that $\angle A$ is either $<$, equal to, or $> \angle D$.

If we suppose $\angle A = \angle D$, $\triangle ABC$ would equal $\triangle DEF$.

[Two ⊿s are equal when two sides and the included $\angle$ of one are equal respectively to two sides and the included $\angle$ of the other.] (§ 63)

Then, BC would equal EF.

[In equal figures, the homologous parts are equal.] (§ 66)

Again, if we suppose $\angle A < \angle D$, BC would be $< EF$.

[If two ⊿s have two sides of one equal respectively to two sides of the other, but the included $\angle$ of the first $>$ the included $\angle$ of the second, the third side of the first is $>$ the third side of the second.] (§ 91)

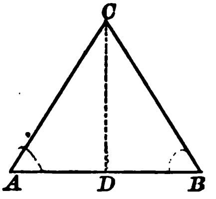
But each of these conclusions is contrary to the hypothesis that BC is $> EF$.

Then, if $\angle A$ can be neither equal to $\angle D$, nor $< \angle D$,

$$\angle A > \angle D.$$

PROP. XXX. THEOREM.

93. *In an isosceles triangle, the angles opposite the equal sides are equal.*



Given AC and BC the equal sides of isosceles $\triangle ABC$.

To Prove $\angle A = \angle B$.

Proof. Draw line $CD \perp AB$.

In rt. ▵ ACD and BCD,

$$CD = CD.$$

And by hyp., $AC = BC$.

$$\therefore \triangle ACD = \triangle BCD.$$

[Two rt. ▵ are equal when the hypotenuse and a leg of one are equal respectively to the hypotenuse and a leg of the other.] (§ 90)

$$\therefore \angle A = \angle B.$$

[In equal figures, the homologous parts are equal.] (§ 66)

94. Cor. I. From equal ▵ ACD and BCD, we have

$AD = BD$, and $\angle ACD = \angle BCD$; hence,

1. *The perpendicular from the vertex to the base of an isosceles triangle bisects the base.*

2. *The perpendicular from the vertex to the base of an isosceles triangle bisects the vertical angle.*

95. Cor. II. *An equilateral triangle is also equiangular.*

PROP. XXXI. THEOREM.

96. (Converse of Prop. XXX.) *If two angles of a triangle are equal, the sides opposite are equal.*

(Fig. of Prop. XXX.)

Given, in $\triangle ABC$, $\angle A = \angle B$.

To Prove $AC = BC$.

(Prove $\triangle ACD = \triangle BCD$ by § 89.)

97. Cor. *An equiangular triangle is also equilateral.*

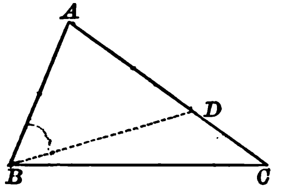
EXERCISES.

22. The angles A and B of a triangle ABC are 57° and 98° respectively; how many degrees are there in the exterior angle at C?

23. How many degrees are there in each angle of an equiangular triangle?

Prop. XXXII. Theorem.

98. *If two sides of a triangle are unequal, the angles opposite are unequal, and the greater angle lies opposite the greater side.*



Given, in $\triangle ABC$, $AC > AB$.

To Prove $\angle ABC > \angle C$.

Proof. Take $AD = AB$, and draw line BD.

Then, in isosceles $\triangle ABD$,

$$\angle ABD = \angle ADB.$$

[In an isosceles $\triangle$, the $\angle$s opposite the equal sides are equal.] (§ 93)

Now since $\angle ADB$ is an ext. $\angle$ of $\triangle BDC$,

$$\angle ADB > \angle C.$$

[An ext. $\angle$ of a $\triangle$ is $>$ either of the opposite int. $\angle$s.] (§ 85)

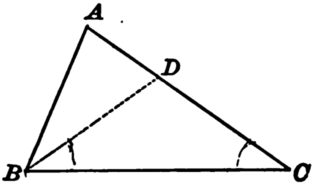
Therefore, its equal, $\angle ABD$, is $> \angle C$.

Then, since $\angle ABC$ is $> \angle ABD$, and $\angle ABD > \angle C$,

$$\angle ABC > \angle C.$$

PROP. XXXIII. THEOREM.

99. (Converse of Prop. XXXII.) *If two angles of a triangle are unequal, the sides opposite are unequal, and the greater side lies opposite the greater angle.*



Given, in $\triangle ABC$, $\angle ABC > \angle C$.

To Prove $AC > AB$.

Proof. Draw line BD, making $\angle CBD = \angle C$, and meeting AC at D.

Then, in $\triangle BCD$, $BD = CD$.

[If two $\angle$s of a $\triangle$ are equal, the sides opposite are equal.] (§ 96)

But, $AD + BD > AB$.

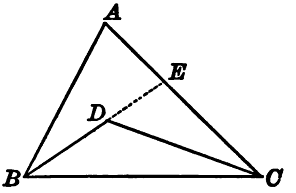
[A str. line is the shortest line between two points.] (Ax. 4)

Substituting for BD its equal CD, we have

$$AD + CD > AB, \text{ or } AC > AB.$$

PROP. XXXIV. THEOREM.

100. *If straight lines be drawn from a point within a triangle to the extremities of any side, the angle included by them is greater than the angle included by the other two sides.*



Given D, any point within $\triangle ABC$, and lines BD and CD.

To Prove $\angle BDC > \angle A$.

Proof. Produce BD to meet AC at E.

Then, since $\angle BDC$ is an ext. $\angle$ of $\triangle CDE$,

$$\angle BDC > \angle DEC.$$

[An ext. $\angle$ of a $\triangle$ is $>$ either of the opposite int. $\angle$s.] (§ 85)

In like manner, since $\angle DEC$ is an ext. $\angle$ of $\triangle ABE$,

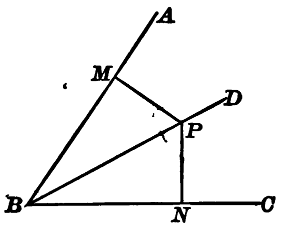
$$\angle DEC > \angle A.$$

Then, since $\angle BDC$ is $> \angle DEC$, and $\angle DEC > \angle A$,

$$\angle BDC > \angle A.$$

PROP. XXXV. THEOREM.

101. *Any point in the bisector of an angle is equally distant from the sides of the angle.*



Given P, any point in bisector BD of $\angle ABC$, and lines PM and $PN \perp$ to AB and BC, respectively.

To Prove $PM = PN$.

Proof. In rt. $\triangle$s BPM and BPN,

$$BP = BP.$$

And by hyp., $\angle PBM = \angle PBN$.

$$\therefore \triangle BPM = \triangle BPN.$$

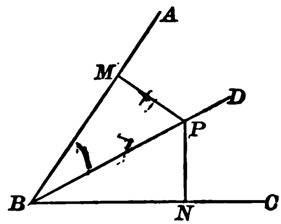
[Two rt. $\triangle$s are equal when the hypotenuse and an adj. $\angle$ of one are equal respectively to the hypotenuse and an adj. $\angle$ of the other.] (§ 70)

$$\therefore PM = PN.$$

[In equal figures, the homologous parts are equal.] (§ 66)

Prop. XXXVI. Theorem.

102. (Converse of Prop. XXXV.) ***Every point which is within an angle, and equally distant from its sides, lies in the bisector of the angle.***



Given point P within $\angle ABC$, equally distant from sides AB and BC, and line BP.

To Prove $\angle PBM = \angle PBN$.

(Prove $\triangle BPM = \triangle BPN$, by § 90; the theorem then follows by § 66.)

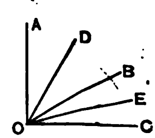
EXERCISES.

24. The angle at the vertex of an isosceles triangle ABC is equal to five-thirds the sum of the equal angles B and C. How many degrees are there in each angle?

25. If from a point O in a straight line AB lines OC and OD be drawn on opposite sides of AB, making $\angle AOC = \angle BOD$, prove that OC and OD lie in the same straight line.

(Fig. of Prop. IV. We have $\angle AOD + \angle BOD = 2$ rt. $\angle$s, and by hyp., $\angle BOD = \angle AOC$.)

26. If the bisectors of two adjacent angles make an angle of 45° with each other, the angles are complementary.



(Given OD and OE the bisectors of $\angle$s AOB and BOC, respectively, and $\angle DOE = 45°$; to prove $\angle$s AOB and BOC complementary.)

27. Prove Prop. XXX. by drawing CD to bisect $\angle ACB$. (§ 63.)

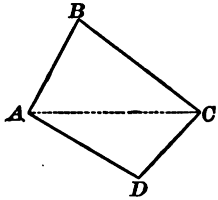
28. Prove Prop. XXX. by drawing CD to the middle point of AB.

29. Prove Prop. XXXI. by drawing CD to bisect $\angle ACB$. (§ 68.)

QUADRILATERALS.

DEFINITIONS.

103. A *quadrilateral* is a portion of a plane bounded by four straight lines; as *ABCD*.



The bounding lines are called the *sides* of the quadrilateral, and their points of intersection the *vertices*.

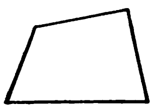
The *angles* of the quadrilateral are the angles included between the adjacent sides.

A *diagonal* is a straight line joining two opposite vertices; as *AC*.

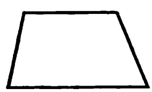
104. A *Trapezium* is a quadrilateral no two of whose sides are parallel.

A *Trapezoid* is a quadrilateral two, and only two, of whose sides are parallel.

A *Parallelogram* (▱) is a quadrilateral whose opposite sides are parallel.



Trapezium.



Trapezoid.

Parallelogram.

The *bases* of a trapezoid are its parallel sides; the *altitude* is the perpendicular distance between them.

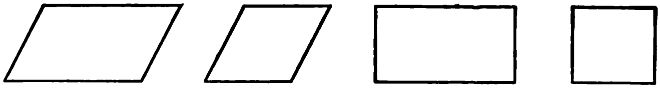
If either pair of parallel sides of a parallelogram be taken and called the *bases*, the *altitude* corresponding to these bases is the perpendicular distance between them.

105. A *Rhomboid* is a parallelogram whose angles are not right angles, and whose adjacent sides are unequal.

A *Rhombus* is a parallelogram whose angles are not right angles, and whose adjacent sides are equal.

A *Rectangle* is a parallelogram whose angles are right angles.

A *Square* is a rectangle whose sides are equal.



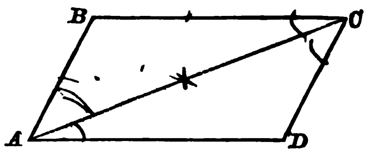
Rhomboid. *Rhombus.* *Rectangle.* *Square.*

PROP. XXXVII. THEOREM.

106. *In any parallelogram,*

I. *The opposite sides are equal.*

II. *The opposite angles are equal.*



Given $\square ABCD$.

I. **To Prove** $AB = CD$ and $BC = AD$.

Proof. Draw diagonal AC.

In $\triangle$s ABC and ACD, $AC = AC$.

Again, since $\parallel$s BC and AD are cut by AC,

$$\angle BCA = \angle CAD.$$

[If two $\parallel$s are cut by a transversal, the alt. int. $\angle$s are equal.] (§ 72)

In like manner, since $\parallel$s AB and CD are cut by AC,

$$\angle BAC = \angle ACD.$$

$$\therefore \triangle ABC = \triangle ACD.$$

[Two $\triangle$s are equal when a side and two adj. $\angle$s of one are equal respectively to a side and two adj. $\angle$s of the other.] (§ 68)

$$\therefore AB = CD \text{ and } BC = AD.$$

[In equal figures, the homologous parts are equal.] (§ 66)

II. **To Prove** $\angle BAD = \angle BCD$ and $\angle B = \angle D$.

Proof. We have $AB \parallel CD$, and $AD \parallel CB$; and AB and CD, and also AD and CB, extend in opposite directions from A and C.

$$\therefore \angle BAD = \angle BCD.$$

[Two $\angle$s whose sides are $\parallel$, each to each, are equal if both pairs of $\parallel$ sides extend in opposite directions from their vertices.] (§ 81)

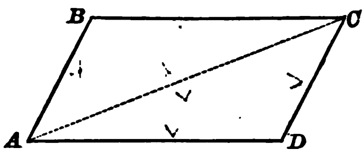
In like manner, $\angle B = \angle D.$

107. Cor. I. *Parallel lines included between parallel lines are equal.*

108. Cor. II. *A diagonal of a parallelogram divides it into two equal triangles.*

Prop. XXXVIII. Theorem.

109. (Converse of Prop. XXXVII, I.) *If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.*



Given, in quadrilateral $ABCD$,

$$AB = CD \text{ and } BC = AD.$$

To Prove $ABCD$ a $\square$.

Proof. Draw diagonal AC.

In $\triangle$s ABC and ACD, $AC = AC$.

And by hyp., $AB = CD$ and $BC = AD$.

$$\therefore \triangle ABC = \triangle ACD.$$

[Two $\triangle$s are equal when the three sides of one are equal respectively to the three sides of the other.] (§ 69)

$$\therefore \angle BCA = \angle CAD \text{ and } \angle BAC = \angle ACD.$$

[In equal figures, the homologous parts are equal.] (§ 66)

Since $\angle BCA = \angle CAD$, $BC \parallel AD$.

[If two str. lines are cut by a transversal, and the alt. int. $\angle$s are equal, the two lines are $\parallel$.] (§ 73)

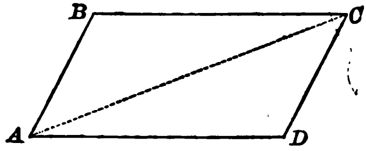
In like manner, since $\angle BAC = \angle ACD$, $AB \parallel CD$.

Then by def., $ABCD$ is a $\square$.

Ex. 30. If one angle of a parallelogram is 119°, how many degrees are there in each of the others?

PROP. XXXIX. THEOREM.

110. *If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.*



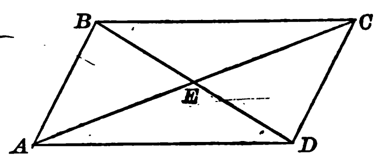
Given, in quadrilateral $ABCD$, BC equal and $\parallel$ to AD.

To Prove $ABCD$ a $\square$.

(Prove $\triangle ABC = \triangle ACD$, by § 63; then, the other two sides of the quadrilateral are equal, and the theorem follows by § 109.)

PROP. XL. THEOREM.

111. *The diagonals of a parallelogram bisect each other.*



Given diagonals AC and BD of $\square$ $ABCD$ intersecting at E.

To Prove $AE = EC$ and $BE = ED$.

(Prove $\triangle AED = \triangle BEC$, by § 68.)

Note. The point E is called the *centre* of the parallelogram.

PROP. XLI. THEOREM.

112. (Converse of Prop. XL.) *If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.*

(Fig. of Prop. XL.)

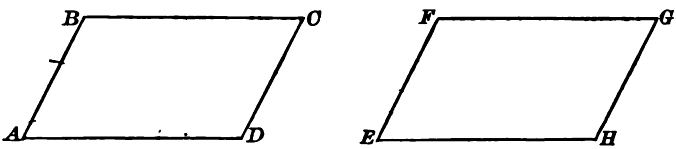
Given AC and BD, the diagonals of quadrilateral $ABCD$, bisecting each other at E.

To Prove $ABCD$ a ▱.

(Prove $\triangle AED = \triangle BEC$, by § 63; then $AD = BC$; in like manner, $AB = CD$, and the theorem follows by § 109.)

Prop. XLII. Theorem.

113. *Two parallelograms are equal when two adjacent sides and the included angle of one are equal respectively to two adjacent sides and the included angle of the other.*



Given, in ▱s $ABCD$ and $EFGH$,

$$AB = EF, AD = EH, \text{ and } \angle A = \angle E.$$

To Prove ▱ $ABCD$ = ▱ $EFGH$.

Proof. Superpose ▱ $ABCD$ upon ▱ $EFGH$ in such a way that $\angle A$ shall coincide with its equal $\angle E$; side AB falling on side EF, and side AD on side EH.

Then since $AB = EF$ and $AD = EH$, point B will fall on point F, and point D on point H.

Now since $BC \parallel AD$ and $FG \parallel EH$, side BC will fall on side FG, and point C will fall somewhere on FG.

[But one str. line can be drawn through a given point ∥ to a given str. line.] (§ 53)

In like manner, side DC will fall on side HG, and point C will fall somewhere on HG.

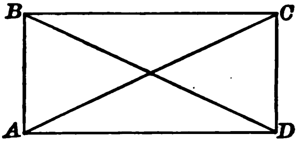
Then point C, falling at the same time on FG and HG, must fall at their intersection G.

Hence, the ▱s coincide throughout, and are equal.

114. Cor. *Two rectangles are equal if the base and altitude of one are equal respectively to the base and altitude of the other.*

Prop. XLIII. Theorem.

115. *The diagonals of a rectangle are equal.*



Given AC and BD the diagonals of rect. $ABCD$.

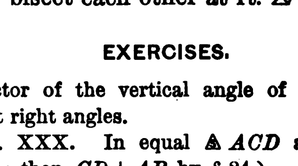
To Prove $AC = BD$.

(Prove rt. $\triangle ABD =$ rt. $\triangle ACD$, by § 63.)

116. Cor. *The diagonals of a square are equal.*

Prop. XLIV. Theorem.

117. *The diagonals of a rhombus bisect each other at right angles.*



(AC and BD bisect each other at rt. $\angle$s by § 43.)

EXERCISES.

31. The bisector of the vertical angle of an isosceles triangle bisects the base at right angles.

(Fig. of Prop. XXX. In equal $\triangle$s ACD and BCD, we have $\angle ADC = \angle BDC$; then $CD \perp AB$ by § 24.)

32. The line joining the vertex of an isosceles triangle to the middle point of the base, is perpendicular to the base, and bisects the vertical angle.

(Fig. of Prop. XXX. Prove $CD \perp AB$ as in Ex. 31.)

33. If one angle of a parallelogram is a right angle, the figure is a rectangle.

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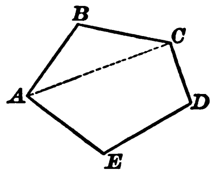
POLYGONS.

DEFINITIONS.

118. A *polygon* is a portion of a plane bounded by three or more straight lines; as $ABCDE$.



The bounding lines are called the *sides* of the polygon, and their sum is called the *perimeter.*

The *angles* of the polygon are the angles EAB, ABC, etc., included between the adjacent sides; and their vertices are called the *vertices* of the polygon.

A *diagonal* of a polygon is a straight line joining any two vertices which are not consecutive; as AC.

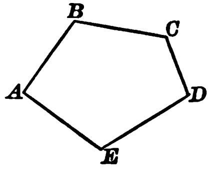
119. Polygons are classified with reference to the number of their sides, as follows:

No. of Sides.	Designation.	No. of Sides.	Designation.
3	Triangle.	8	Octagon.
4	Quadrilateral.	9	Enneagon.
5	Pentagon.	10	Decagon.
6	Hexagon.	11	Hendecagon.
7	Heptagon.	12	Dodecagon.

120. An *equilateral* polygon is a polygon all of whose sides are equal.

An *equiangular* polygon is a polygon all of whose angles are equal.

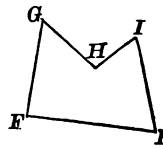
121. A polygon is called *convex* when no side, if produced, will enter the surface enclosed by the perimeter; as $ABCDE$.



It is evident that, in such a case, each angle of the polygon is less than two right angles.

All polygons considered hereafter will be understood to be convex, unless the contrary is stated.

A polygon is called *concave* when at least two of its sides, if produced, will enter the surface enclosed by the perimeter; as *FGHIK*.

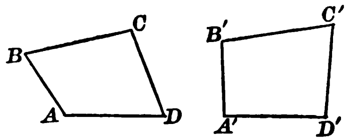


It is evident that, in such a case, at least one angle of the polygon is greater than two right angles.

Thus, in polygon *FGHIK*, the interior angle *GHI* is greater than two right angles.

Such an angle is called *re-entrant*.

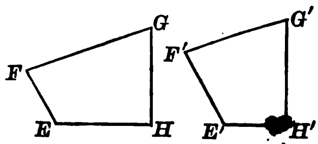
122. Two polygons are said to be ***mutually equilateral*** when the sides of one are equal respectively to the sides of the other, when taken in the same order.



Thus, polygons *ABCD* and $A'B'C'D'$ are mutually equilateral if

$$AB = A'B',\ BC = B'C',\ CD = C'D',\ \text{and}\ DA = D'A'.$$

Two polygons are said to be ***mutually equiangular*** when the angles of one are equal respectively to the angles of the other when taken in the same order.



Thus, polygons *EFGH* and $E'F'G'H'$ are mutually equiangular if

$$\angle E = \angle E',\ \angle F = \angle F',\ \angle G = \angle G',\ \text{and}\ \angle H = \angle H'.$$

123. In polygons which are mutually equilateral or mutually equiangular, sides or angles which are similarly placed are called *homologous*.

In mutually equiangular polygons, the sides included between equal angles are homologous.

124. If two *triangles* are mutually equilateral, they are also mutually equiangular (§ 69).

But with this exception, two polygons may be mutually equilateral without being mutually equiangular, or mutually equiangular without being mutually equilateral.

If two polygons are both mutually equilateral and mutually equiangular, they are equal.

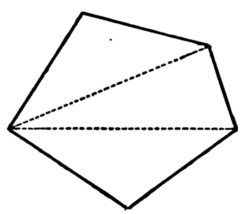
For they can evidently be applied one to the other so as to coincide throughout.

125. *Two polygons are equal when they are composed of the same number of triangles, equal each to each, and similarly placed.*

For they can evidently be applied one to the other so as to coincide throughout.

Prop. XLV. Theorem.

126. *The sum of the angles of any polygon is equal to two right angles taken as many times, less two, as the polygon has sides.*



Given a polygon of n sides.

To Prove the sum of its $\angle$s equal to $n-2$ times two rt. $\angle$s.

Proof. The polygon may be divided into $n-2$ $\triangle$s by drawing diagonals from one of its vertices.

The sum of the $\angle$s of the polygon is equal to the sum of the $\angle$s of the $\triangle$s.

But the sum of the $\angle$s of each $\triangle$ is two rt. $\angle$s.

[The sum of the $\angle$s of any $\triangle$ is equal to two rt. $\angle$s.] (§ 84)

Hence, the sum of the $\angle$s of the polygon is $n-2$ times two rt. $\angle$s.

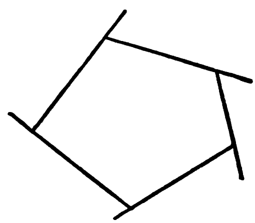
127. Cor. I. *The sum of the angles of any polygon is equal to twice as many right angles as the polygon has sides, less four right angles.*

For if R represents a rt. ∠, and n the number of sides of a polygon, the sum of its ∠s is $(n-2) \times 2\,R$, or $2\,nR - 4\,R$.

128. Cor. II. *The sum of the angles of a quadrilateral is equal to four right angles; of a pentagon, six right angles; of a hexagon, eight right angles; etc.*

Prop. XLVI. Theorem.

129. *If the sides of any polygon be produced so as to make an exterior angle at each vertex, the sum of these exterior angles is equal to four right angles.*



Given a polygon of n sides with its sides produced so as to make an ext. ∠ at each vertex.

To Prove the sum of these ext. ∠s equal to 4 rt. ∠s.

Proof. The sum of the ext. and int. ∠s at any one vertex is two rt. ∠s.

[If two adj. ∠s have their ext. sides in the same str. line, their sum is equal to two rt. ∠s.] (§ 32)

Hence, the sum of *all* the ext. and int. ∠s is $2\,n$ rt. ∠s.
But the sum of the int. ∠s alone is $2\,n$ rt. ∠s − 4 rt. ∠s.

[The sum of the ∠s of any polygon is equal to twice as many rt. ∠s as the polygon has sides, less 4 rt. ∠s.] (§ 127)

Whence, the sum of the ext. ∠s is 4 rt. ∠s.

EXERCISES.

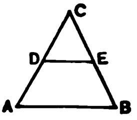
34. How many degrees are there in each angle of an equiangular hexagon? of an equiangular octagon? of an equiangular decagon? of an equiangular dodecagon?

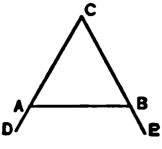
35. How many degrees are there in the exterior angle at each vertex of an equiangular pentagon?

36. If two angles of a quadrilateral are supplementary, the other two angles are supplementary.

37. If, in a triangle ABC, $\angle A = \angle B$, a line parallel to AB makes equal angles with sides AC and BC.

(To prove $\angle CDE = \angle CED$.)

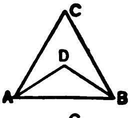




38. If the equal sides of an isosceles triangle be produced, the exterior angles made with the base are equal. (§ 31, 2.)

39. If the perpendicular from the vertex to the base of a triangle bisects the base, the triangle is isosceles.

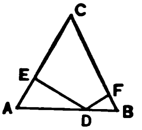
(Fig. of Prop. XXX. ▵ ACD and BCD are equal by § 63.)



40. The bisectors of the equal angles of an isosceles triangle form, with the base, another isosceles triangle.

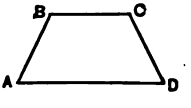
41. If from any point in the base of an isosceles triangle perpendiculars to the equal sides be drawn, they make equal angles with the base.

($\angle ADE = \angle BDF$, by § 31, 1.)



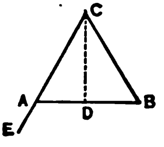
42. If the angles adjacent to one base of a trapezoid are equal, those adjacent to the other base are also equal.

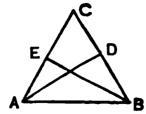
(Given $\angle A = \angle D$; to prove $\angle B = \angle C$.)



43. Either exterior angle at the base of an isosceles triangle is equal to the sum of a right angle and one-half the vertical angle.

($\angle DAE$ is an ext. $\angle$ of $\triangle ACD$.)



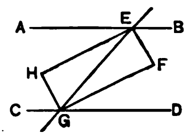


44. The straight lines bisecting the equal angles of an isosceles triangle, and terminating in the opposite sides, are equal.

($\triangle ABD = \triangle ABE$.)

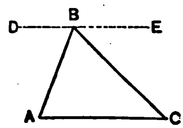
45. Two isosceles triangles are equal when the base and vertical angle of one are equal respectively to the base and vertical angle of the other.

(Each of the remaining $\angle$s of one $\triangle$ is equal to each of the remaining $\angle$s of the other.)



46. If two parallels are cut by a transversal, the bisectors of the four interior angles form a rectangle.

($EH \parallel FG$, by § 73; in like manner, $EF \parallel GH$; then use Exs. 12 and 33.)



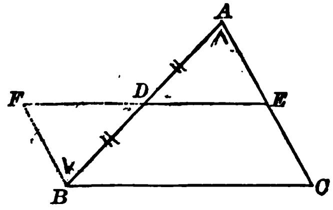
47. Prove Prop. XXVI. by drawing through B a line parallel to AC.

(Sum of $\angle$s at $B = 2$ rt. $\angle$s.)

MISCELLANEOUS THEOREMS.

Prop. XLVII. Theorem.

130. *The line joining the middle points of two sides of a triangle is parallel to the third side, and equal to one-half of it.*



Given line DE joining middle points of sides AB and AC, respectively, of $\triangle ABC$.

To Prove $DE \parallel BC$, and $DE = \frac{1}{2} BC$.

Proof. Draw line $BF \parallel AC$, meeting ED produced at F.

In $\triangle$ ADE and BDF,

$$\angle ADE = \angle BDF.$$

[If two str. lines intersect, the vertical $\angle$s are equal.] (§ 40)

Also, since ||s AC and BF are cut by AB,

$$\angle A = \angle DBF.$$

[If two ||s are cut by a transversal, the alt. int. $\angle$s are equal.] (§ 72)

And by hyp., $AD = BD$.

$$\therefore \triangle ADE = \triangle BDF.$$

[Two $\triangle$s are equal when a side and two adj. $\angle$s of one are equal respectively to a side and two adj. $\angle$s of the other.] (§ 68)

$$\therefore DE = DF \text{ and } AE = BF.$$

[In equal figures, the homologous parts are equal.] (§ 66)

Then since, by hyp., $AE = EC$, BF is equal and || to CE.

Whence, $BCEF$ is a $\square$.

[If two sides of a quadrilateral are equal and ||, the figure is a $\square$.] (§ 110)

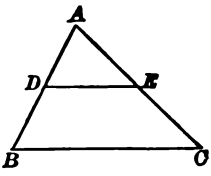
$$\therefore DE \parallel BC.$$

Again, since $DE = DF$,

$$DE = \tfrac{1}{2} FE = \tfrac{1}{2} BC.$$

[In any $\square$, the opposite sides are equal.] (§ 106)

131. Cor. *The line which bisects one side of a triangle, and is parallel to another side, bisects also the third side.*



Given, in $\triangle ABC$, D the middle point of side AB, and line $DE \parallel BC$.

To Prove that DE bisects AC.

Proof. A line joining D to the middle point of AC will be $\parallel BC$.

[The line joining the middle points of two sides of a $\triangle$ is || to the third side.] (§ 130)

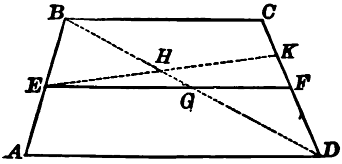
Then this line will coincide with DE.

[But one str. line can be drawn through a given point || to a given str. line.] (§ 53)

Therefore, DE bisects AC.

PROP. XLVIII. THEOREM.

132. *The line joining the middle points of the non-parallel sides of a trapezoid is parallel to the bases, and equal to one-half their sum.*



Given line EF joining middle points of non-$\parallel$ sides AB and CD, respectively, of trapezoid $ABCD$.

To Prove $EF \parallel$ to AD and BC, and $EF = \frac{1}{2}(AD + BC)$.

Proof. If EF is not $\parallel$ to AD and BC, draw line EK $\parallel$ to AD and BC, meeting CD at K; and draw line BD intersecting EF at G, and EK at H.

In $\triangle ABD$, EH is $\parallel AD$ and bisects AB; then it bisects BD.

[The line which bisects one side of a $\triangle$, and is $\parallel$ to another side, bisects also the third side.] (§ 131)

In like manner, in $\triangle BCD$, HK is $\parallel BC$ and bisects BD; then it bisects CD.

But this is impossible unless EK coincides with EF.

[But one str. line can be drawn between two points.] (Ax. 3)

Hence, EF is $\parallel$ to AD and BC.

Again, since EG coincides with EH, and EH bisects AB and BD,

$$EG = \tfrac{1}{2} AD. \tag{1}$$

[The line joining the middle points of two sides of a $\triangle$ is equal to one-half the third side.] (§ 130)

In like manner, since GF bisects BD and CD,

$$GF = \tfrac{1}{2} BC. \tag{2}$$

Adding (1) and (2),

$$EG + GF = \tfrac{1}{2} AD + \tfrac{1}{2} BC.$$

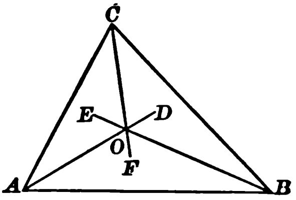
Or,

$$EF = \tfrac{1}{2}(AD + BC).$$

133. Cor. *The line which is parallel to the bases of a trapezoid, and bisects one of the non-parallel sides, bisects the other also.*

PROP. XLIX. THEOREM.

134. *The bisectors of the angles of a triangle intersect at a common point.*



Given lines AD, BE, and CF bisecting $\angle s\ A$, B, and C, respectively, of $\triangle ABC$.

To Prove that AD, BE, and CF intersect at a common point.

Proof. Let AD and BE intersect at O.

Since O is in bisector AD, it is equally distant from sides AB and AC.

[Any point in the bisector of an $\angle$ is equally distant from the sides of the $\angle$.] (§ 101)

In like manner, since O is in bisector BE, it is equally distant from sides AB and BC.

Then O is equally distant from sides AC and BC, and therefore lies in bisector CF.

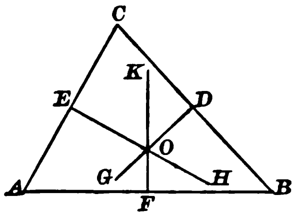
[Every point which is within an $\angle$, and equally distant from its sides, lies in the bisector of the $\angle$.] (§ 102)

Hence, AD, BE, and CF intersect at the common point O.

135. Cor. *The point of intersection of the bisectors of the angles of a triangle is equally distant from the sides of the triangle.*

Prop. L. Theorem.

136. *The perpendiculars erected at the middle points of the sides of a triangle intersect at a common point.*



Given DG, EH, and FK the ⊥s erected at middle points D, E, and F, of sides BC, CA, and AB, respectively, of $\triangle ABC$.

To Prove that DG, EH, and FK intersect at a common point.

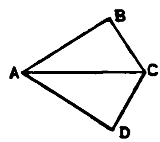
(Let DG and EH intersect at O; by § 41, O is equally distant from B and C; it is also equally distant from A and C; the theorem follows by § 42.)

137. Cor. *The point of intersection of the perpendiculars erected at the middle points of the sides of a triangle, is equally distant from the vertices of the triangle.*

EXERCISES.

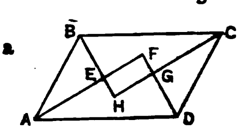
48. If the diagonals of a parallelogram are equal, the figure is a rectangle.

(Fig. of Prop. XLIII. △s ABD and ACD are equal, and therefore $\angle BAD = \angle ADC$; also, these △s are supplementary.)



49. If two adjacent sides of a quadrilateral are equal, and the diagonal bisects their included angle, the other two sides are equal.

(Given $AB = AD$, and AC bisecting $\angle BAD$; to prove $BC = CD$.)

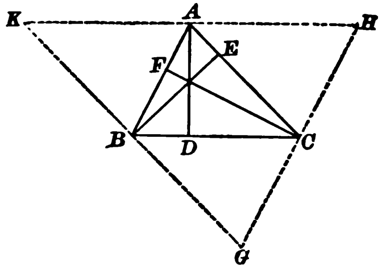


50. The bisectors of the interior angles of a parallelogram form a rectangle.

(By Ex. 46, each $\angle$ of $EFGH$ is a rt. $\angle$.)

PROP. LI. THEOREM.

138. *The perpendiculars from the vertices of a triangle to the opposite sides intersect at a common point.*



Given AD, BE, and CF the ⊥s from the vertices of $\triangle ABC$ to the opposite sides.

To Prove that AD, BE, and CF intersect at a common point.

Proof. Through A, B, and C, draw lines HK, KG, and GH ∥ to BC, CA, and AB, respectively, forming $\triangle GHK$.

Then AD, being $\perp BC$, is also $\perp HK$.

[A str. line ⊥ to one of two ∥s is ⊥ to the other.] (§ 56)

Now since, by cons., $ABCH$ and $ACBK$ are ▱s,

$$AH = BC \text{ and } AK = BC.$$

[In any ▱, the opposite sides are equal.] (§ 106)

$$\therefore AH = AK.$$

[Things which are equal to the same thing, are equal to each other.] (Ax. 1)

Then AD is $\perp HK$ at the middle point of HK.

In like manner, BE and CF are ⊥ to KG and GH, respectively, at their middle points.

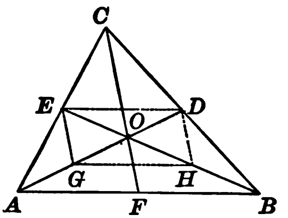
Then, AD, BE, and CF being ⊥ to the sides of $\triangle GHK$ at their middle points, intersect at a common point.

[The ⊥ erected at the middle points of the sides of a △ intersect at a common point.] (§ 136)

139. Def. A *median* of a triangle is a line drawn from any vertex to the middle point of the opposite side.

PROP. LII. THEOREM.

140. *The medians of a triangle intersect at a common point, which lies two-thirds the way from each vertex to the middle point of the opposite side.*



Given AD, BE, and CF the medians of $\triangle ABC$.

To Prove that AD, BE, and CF intersect at a common point, which lies two-thirds the way from each vertex to the middle point of the opposite side.

Proof. Let AD and BE intersect at O.

Let G and H be the middle points of OA and OB, respectively, and draw lines ED, GH, EG, and DH.

Since ED bisects AC and BC,

$$\therefore ED \parallel AB \text{ and } = \tfrac{1}{2} AB.$$

[The line joining the middle points of two sides of a $\triangle$ is $\parallel$ to the third side, and equal to one-half of it.] (§ 130)

In like manner, since GH bisects OA and OB,

$$GH \parallel AB \text{ and } = \tfrac{1}{2} AB.$$

Then ED and GH are equal and $\parallel$.

[Things which are equal to the same thing, are equal to each other.] (Ax. 1)

[Two str. lines $\parallel$ to the same str. line are $\parallel$ to each other.] (§ 55)

Therefore, $EDHG$ is a ▱.

[If two sides of a quadrilateral are equal and $\parallel$, the figure is a ▱.] (§ 110)

Then GD and EH bisect each other at O.

[The diagonals of a ▱ bisect each other.] (§ 111)

But by hyp., G is the middle point of OA, and H of OB.

$$\therefore\ AG = OG = OD, \text{ and } BH = OH = OE.$$

That is, AD and BE intersect at a point O which lies two-thirds the way from A to D, and from B to E.

In like manner, AD and CF intersect at a point which lies two-thirds the way from A to D, and from C to F.

Hence, AD, BE, and CF intersect at the common point O, which lies two-thirds the way from each vertex to the middle point of the opposite side.

LOCI.

141. Def. If a series of points, all of which satisfy a certain condition, lie in a certain line, and every point in this line satisfies the given condition, the line is said to be the *locus* of the points.

For example, every point which satisfies the condition of being equally distant from the extremities of a straight line, lies in the perpendicular erected at the middle point of the line (§ 42).

Also, every point in the perpendicular erected at the middle point of a line satisfies the condition of being equally distant from the extremities of the line (§ 41).

Hence, *the perpendicular erected at the middle point of a straight line is the* **Locus** *of points which are equally distant from the extremities of the line.*

Again, every point which satisfies the condition of being within an angle, and equally distant from its sides, lies in the bisector of the angle (§ 102).

Also, every point in the bisector of an angle satisfies the condition of being equally distant from its sides (§ 101).

Hence, *the bisector of an angle is the locus of points which are within the angle, and equally distant from its sides.*

EXERCISES.

51. Two straight lines are parallel if any two points of either are equally distant from the other.

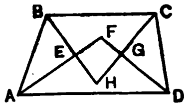
(Prove by *Reductio ad Absurdum.*)

52. What is the locus of points at a given distance from a given straight line? (Ex. 51.)

53. What is the locus of points equally distant from a pair of intersecting straight lines?

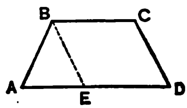
54. What is the locus of points equally distant from a pair of parallel straight lines?

55. The bisectors of the interior angles of a trapezoid form a quadrilateral, two of whose angles are right angles. (Ex. 46.)



56. If the angles at the base of a trapezoid are equal, the non-parallel sides are also equal.

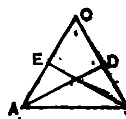
(Given $\angle A = \angle D$; to prove $AB = CD$. Draw $BE \parallel CD$.)



57. If the non-parallel sides of a trapezoid are equal, the angles which they make with the bases are equal.

(Fig. of Ex. 56. Given $AB = CD$; to prove $\angle A = \angle D$, and also $\angle ABC = \angle C$. Draw $BE \parallel CD$.)

58. The perpendiculars from the extremities of the base of an isosceles triangle to the opposite sides are equal.

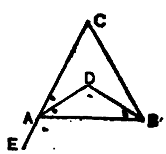


59. If the perpendiculars from the extremities of the base of a triangle to the opposite sides are equal, the triangle is isosceles.

(Converse of Ex. 58. Prove $\triangle ACD = \triangle BCE$.)

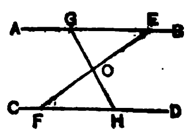
60. The angle between the bisectors of the equal angles of an isosceles triangle is equal to the exterior angle at the base of the triangle.

($\angle ADB = 180° - (\angle BAD + \angle ABD)$.)



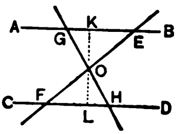
61. If a line joining two parallels be bisected, any line drawn through the point of bisection and included between the parallels will be bisected at the point.

(To prove that GH is bisected at O.)

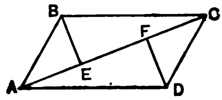


62. If through a point midway between two parallels two transversals be drawn, they intercept equal portions of the parallels.

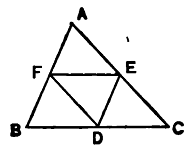
(Draw $OK \perp AB$, and produce KO to meet CD at L. Then $\triangle\ OGK = \triangle\ OHL$.)



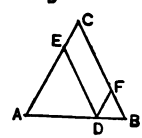
63. If perpendiculars BE and DF be drawn from vertices B and D of parallelogram $ABCD$ to the diagonal AC, prove $BE = DF$. (§ 70.)



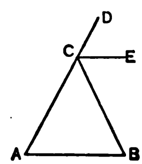
64. The lines joining the middle points of the sides of a triangle divide it into four equal triangles. (§ 130.)



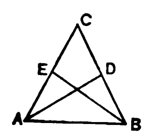
65. If from any point in the base of an isosceles triangle parallels to the equal sides be drawn, the perimeter of the parallelogram formed is equal to the sum of the equal sides of the triangle. (§ 96.)



66. The bisector of the exterior angle at the vertex of an isosceles triangle is parallel to the base. (§ 85, 1.)

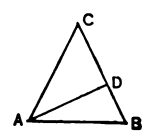


67. The medians drawn from the extremities of the base of an isosceles triangle are equal.



68. If from the vertex of one of the equal angles of an isosceles triangle a perpendicular be drawn to the opposite side, it makes with the base an angle equal to one-half the vertical angle of the triangle.

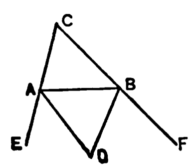
(To prove $\angle BAD = \frac{1}{2} \angle C$.)



69. If the exterior angles at the vertices A and B of triangle ABC are bisected by lines which meet at D, prove

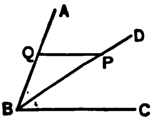
$$\angle ADB = 90^\circ - \tfrac{1}{2} C.$$

$$(\angle ADB = 180^\circ - (\angle BAD + \angle ABD).)$$



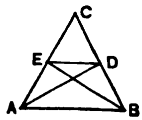
70. The diagonals of a rhombus bisect its angles. (Fig. of Prop. XLIV.)

71. If from any point in the bisector of an angle a parallel to one of the sides be drawn, the bisector, the parallel, and the remaining side form an isosceles triangle.

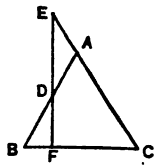


72. If the bisectors of the equal angles of an isosceles triangle meet the equal sides at D and E, prove DE parallel to the base of the triangle.

(Prove $\triangle$ CED isosceles.)

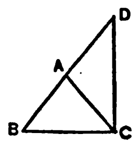


73. If at any point D in one of the equal sides AB of isosceles triangle ABC, DE be drawn perpendicular to base BC meeting CA produced at E, prove triangle ADE isosceles.

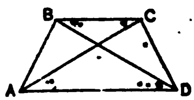


74. From C, one of the extremities of the base BC of isosceles triangle ABC, a line is drawn meeting BA produced at D, making $AD = AB$. Prove CD perpendicular to BC. (§ 84.)

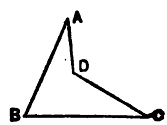
($\triangle$ ACD is isosceles.)



75. If the non-parallel sides of a trapezoid are equal, its diagonals are also equal. (Ex. 57.)

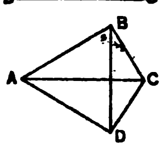


76. If ADC is a re-entrant angle of quadrilateral $ABCD$, prove that angle ADC, exterior to the figure, is equal to the sum of interior angles A, B, and C. (§ 128.)



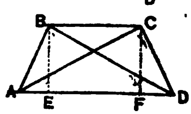
77. If a diagonal of a quadrilateral bisects two of its angles, it is perpendicular to the other diagonal.

(Prove $AC \perp DB$, by § 43.)



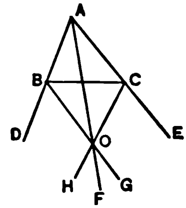
78. In a quadrilateral $ABCD$, angles ABD and CAD are equal to ACD and BDA, respectively; prove BC parallel to AD.

(Prove $AB = CD$; then prove $BE = CF$.)



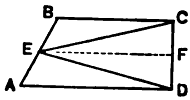
79. State and prove the converse of Prop. XLIV. (§ 41, I.)

80. State and prove the converse of Ex. 66, p. 67. (§ 96.)



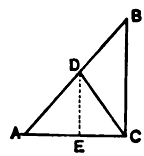
81. The bisectors of the exterior angles at two vertices of a triangle, and the bisector of the interior angle at the third vertex meet at a common point.

(Prove as in § 134.)



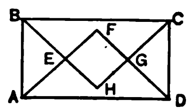
82. $ABCD$ is a trapezoid whose parallel sides AD and BC are perpendicular to CD. If E is the middle point of AB, prove $EC = ED$. (§ 41, I.)

(Draw $EF \parallel AD$.)



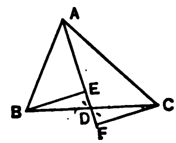
83. The middle point of the hypotenuse of a right triangle is equally distant from the vertices of the triangle.

(To prove $AD = BD = CD$. Draw $DE \parallel BC$.)

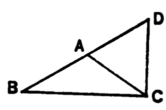


84. The bisectors of the angles of a rectangle form a square.

(By Ex. 50, $EFGH$ is a rectangle. Now prove $AF = BH$ and $AE = BE$.)



85. If D is the middle point of side BC of triangle ABC, and BE and CF are perpendiculars from B and C to AD, produced if necessary, prove $BE = CF$.



86. The angle at the vertex of isosceles triangle ABC is equal to twice the sum of the equal angles B and C. If CD be drawn perpendicular to BC, meeting BA produced at D, prove triangle ACD equilateral.

(Prove each $\angle$ of $\triangle ACD$ equal to 60°.)

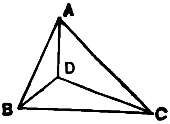
87. If angle B of triangle ABC is greater than angle C, and BD be drawn to AC making $AD = AB$, prove

$$\angle ADB = \tfrac{1}{2}(B + C), \text{ and } \angle CBD = \tfrac{1}{2}(B - C).$$

(Fig. of Prop. XXXII.)

88. How many sides are there in the polygon the sum of whose interior angles exceeds the sum of its exterior angles by 540°?

89. The sum of the lines drawn from any point within a triangle to the vertices is greater than the half-sum of the three sides.

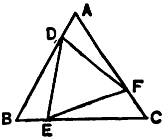


(Apply § 61 to each of the ▲ ABD, ACD, and BCD.)

90. The sum of the lines drawn from any point within a triangle to the vertices is less than the sum of the three sides. (§ 48.)

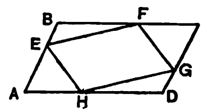
(Fig. of Ex. 89.)

91. If D, E, and F are points on the sides AB, BC, and CA, respectively, of equilateral triangle ABC, such that $AD = BE = CF$, prove DEF an equilateral triangle.

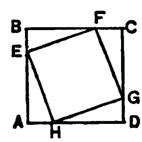


(Prove ▲ ADF, BDE, and CEF equal.)

92. If E, F, G, and H are points on the sides AB, BC, CD, and DA, respectively, of parallelogram $ABCD$, such that $AE = CG$ and $BF = DH$, prove $EFGH$ a parallelogram.

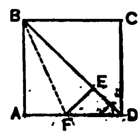


93. If E, F, G, and H are points on sides AB, BC, CD, and DA, respectively, of square $ABCD$, such that $AE = BF = CG = DH$, prove $EFGH$ a square.

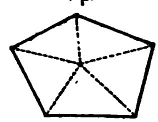


(First prove $EFGH$ equilateral. Then prove $\angle FEH = 90°$.)

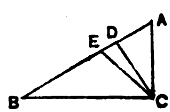
94. If on the diagonal BD of square $ABCD$ a distance BE be taken equal to AB, and EF be drawn perpendicular to BD, meeting AD at F, prove that $AF = EF = ED$.



95. Prove the theorem of § 127 by drawing lines from any point within the polygon to the vertices. (§ 35.)



96. If CD is the perpendicular from the vertex of the right angle to the hypotenuse of right triangle ABC, and CE the bisector of angle C, meeting AB at E, prove $\angle DCE$ equal to one-half the difference of angles A and B.



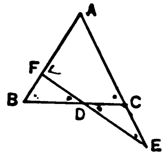
(To prove $\angle DCE = \frac{1}{2}(\angle A - \angle B)$.)

97. State and prove the converse of Ex. 70, p. 68.

(Fig. of Prop. XLIV. Prove the sides all equal.)

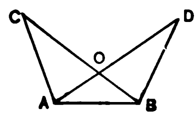
98. State and prove the converse of Ex. 75, p. 68.

(Fig. of Ex. 78. Prove $\triangle$s ACF and BDE equal.)

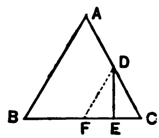


99. D is any point in base BC of isosceles triangle ABC. The side AC is produced from C to E, so that $CE = CD$, and DE is drawn meeting AB at F. Prove $\angle AFE = 3\angle AEF$.

($\angle AFE$ is an ext. $\angle$ of $\triangle BFD$.)

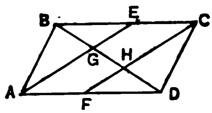


100. If ABC and ABD are two triangles on the same base and on the same side of it, such that $AC = BD$ and $AD = BC$, and AD and BC intersect at O, prove triangle OAB isosceles.



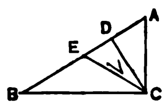
101. If D is the middle point of side AC of equilateral triangle ABC, and DE be drawn perpendicular to BC, prove $EC = \frac{1}{4} BC$.

(Draw DF to the middle point of BC.)



102. If in parallelogram $ABCD$, E and F are the middle points of sides BC and AD, respectively, prove that lines AE and CF trisect diagonal BD.

(By § 131, AE bisects BH, and CF bisects DG.)



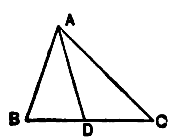
103. If CD is the perpendicular from C to the hypotenuse of right triangle ABC, and E is the middle point of AB, prove $\angle DCE$ equal to the difference of angles A and B. (Ex. 83.)

104. If one acute angle of a right triangle is double the other, the hypotenuse is double the shorter leg.

(Fig. of Ex. 86. Draw CA to middle point of BD.)

105. If AC be drawn from the vertex of the right angle to the hypotenuse of right triangle BCD so as to make $\angle ACD = \angle D$, it bisects the hypotenuse.

(Fig. of Ex. 74. Prove $\triangle ABC$ isosceles.)



106. If D is the middle point of side BC of triangle ABC, prove $AD > \frac{1}{2}(AB + AC - BC)$. (§ 62.)

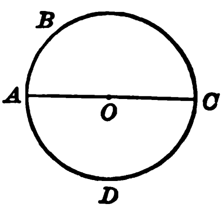
Note. For additional exercises on Book I., see p. 220.

Book II.

THE CIRCLE.

DEFINITIONS.

142. A *circle* (⊙) is a portion of a plane bounded by a curve called a *circumference*, all points of which are equally distant from a point within, called the *centre;* as *ABCD*.



An *arc* is any portion of the circumference; as *AB*.

A *radius* is a straight line drawn from the centre to the circumference; as *OA*.

A *diameter* is a straight line drawn through the centre, having its extremities in the circumference; as *AC*.

143. It follows from the definition of § 142 that

All radii of a circle are equal.

Also, all its diameters are equal, since each is the sum of two radii.

144. *Two circles are equal when their radii are equal.*

For they can evidently be applied one to the other so that their circumferences shall coincide throughout.

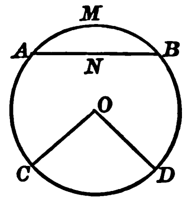
145. Conversely, *the radii of equal circles are equal.*

146. A *semi-circumference* is an arc equal to one-half the circumference.

A *quadrant* is an arc equal to one-fourth the circumference.

Concentric circles are circles having the same centre.

147. A *chord* is a straight line joining the extremities of an arc; as AB.



The arc is said to be *subtended* by its chord.

Every chord subtends two arcs; thus chord AB subtends arcs AMB and $ACDB$.

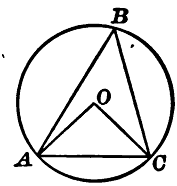
When the arc subtended by a chord is spoken of, that arc which is less than a semi-circumference is understood, unless the contrary is specified.

A *segment* of a circle is the portion included between an arc and its chord; as $AMBN$.

A *semicircle* is a segment equal to one-half the circle.

A *sector* of a circle is the portion included between an arc and the radii drawn to its extremities; as OCD.

148. A *central angle* is an angle whose vertex is at the centre, and whose sides are radii; as AOC.

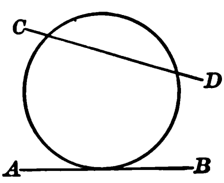


An *inscribed angle* is an angle whose vertex is on the circumference, and whose sides are chords; as ABC.

An angle is said to be *inscribed in a segment* when its vertex is on the arc of the segment, and its sides pass through the extremities of the subtending chord.

Thus, angle B is inscribed in segment ABC.

149. A straight line is said to be *tangent to*, or *touch, a circle* when it has but one point in common with the circumference; as AB.



In such a case, the circle is said to be tangent to the straight line.

The common point is called the *point of contact*, or *point of tangency.*

A *secant* is a straight line which intersects the circumference in two points; as CD.

150. Two circles are said to be *tangent to each other* when they are both tangent to the same straight line at the same point.

They are said to be tangent *internally* or *externally* according as one circle lies entirely within or entirely without the other.

A *common tangent* to two circles is a straight line which is tangent to both of them.

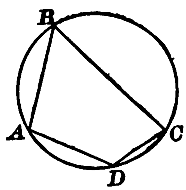
151. A polygon is said to be *inscribed in a circle* when all its vertices lie on the circumference; as *ABCD*.

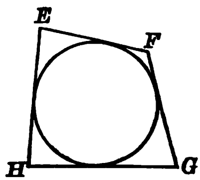
In such a case, the circle is said to be *circumscribed about the polygon.*

A polygon is said to be *inscriptible* when it can be inscribed in a circle.

A polygon is said to be *circumscribed about a circle* when all its sides are tangent to the circle; as *EFGH*.

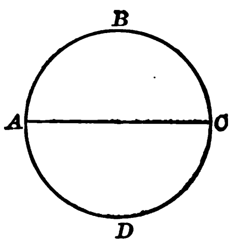
In such a case, the circle is said to be *inscribed in the polygon.*





PROP. I. THEOREM.

152. *Every diameter bisects the circle and its circumference.*



Given *AC* a diameter of ⊙ *ABCD*.

To Prove that *AC* bisects the ⊙, and its circumference.

Proof. Superpose segment ABC upon segment ADC, by folding it over about AC as an axis.

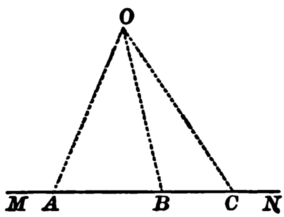
Then, arc ABC will coincide with arc ADC; for otherwise there would be points of the circumference unequally distant from the centre.

Hence, segments ABC and ADC coincide throughout, and are equal.

Therefore, AC bisects the ⊙, and its circumference.

PROP. II. THEOREM.

153. ***A straight line cannot intersect a circumference at more than two points.***



Given O the centre of a ⊙, and MN any str. line.

To Prove that MN cannot intersect the circumference at more than two points.

Proof. If possible, let MN intersect the circumference at three points, A, B, and C; draw radii OA, OB, and OC.

Then, $OA = OB = OC$. (§ 143)

We should then have three equal str. lines drawn from a point to a str. line.

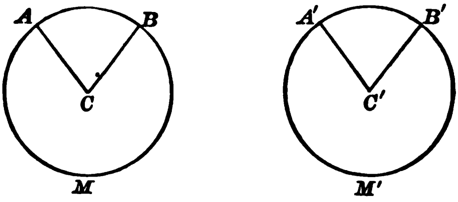
But this is impossible; for it follows from § 49 that not more than two equal str. lines can be drawn from a point to a str. line.

Hence, MN cannot intersect the circumference at more than two points.

Ex. 1. What is the locus of points at a given distance from a given point?

Prop. III. Theorem.

154. *In equal circles, or in the same circle, equal central angles intercept equal arcs on the circumference.*



Given ACB and $A'C'B'$ equal central ∡ of equal ⓢ AMB and $A'M'B'$, respectively.

To Prove arc $AB =$ arc $A'B'$.

Proof. Superpose sector ABC upon sector $A'B'C'$ in such a way that $\angle C$ shall coincide with its equal $\angle C'$.

Now, $AC = A'C'$ and $BC = B'C'$. (§ 145)

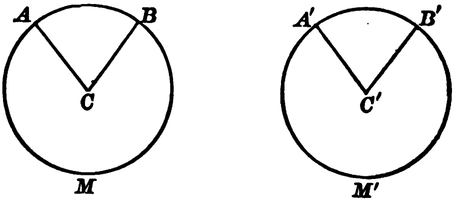
Whence, point A will fall at A', and point B at B'.

Then, arc AB will coincide with arc $A'B'$; for all points of either are equally distant from the centre.

$\therefore$ arc $AB =$ arc $A'B'$.

Prop. IV. Theorem.

155. (Converse of Prop. III.) *In equal circles, or in the same circle, equal arcs are intercepted by equal central angles.*



Given ACB and $A'C'B'$ central ∡ of equal ⓢ AMB and $A'M'B'$, respectively, and arc $AB =$ arc $A'B'$.

To Prove $\angle C = \angle C'$.

Proof. Since the ⓢ are equal, we may superpose $\odot AMB$ upon $\odot A'M'B'$ in such a way that point A shall fall at A', and centre C at C'.

Then since arc $AB =$ arc $A'B'$, point B will fall at B'.

Whence, radii AC and BC will coincide with radii $A'C'$ and $B'C'$, respectively. (Ax. 3)

Hence, $\angle C$ will coincide with $\angle C'$.

$$\therefore \angle C = \angle C'.$$

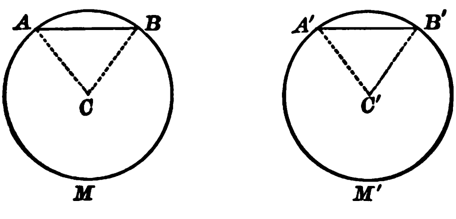
156. Sch. *In equal circles, or in the same circle,*

1. *The greater of two central angles intercepts the greater arc on the circumference.*

2. *The greater of two arcs is intercepted by the greater central angle.*

PROP. V. THEOREM.

157. *In equal circles, or in the same circle, equal chords subtend equal arcs.*



Given, in equal ⓢ AMB and $A'M'B'$,
chord $AB =$ chord $A'B'$.

To Prove arc $AB =$ arc $A'B'$.

Proof. Draw radii AC, BC, $A'C'$, and $B'C'$.

Then in ⚠ ABC and $A'B'C'$, by hyp.,

$$AB = A'B'.$$

Also, $AC = A'C'$ and $BC = B'C'$. (?)

$\therefore \triangle ABC = \triangle A'B'C'$. (?)

$\therefore \angle C = \angle C'$. (?)

$\therefore$ arc $AB =$ arc $A'B'$. (§ 154)

PROP. VI. THEOREM.

158. (Converse of Prop. V.) *In equal circles, or in the same circle, equal arcs are subtended by equal chords.*

(Fig. of Prop. V.)

Given, in equal $\circledS$ AMB and $A'M'B'$, arc $AB =$ arc $A'B'$; and chords AB and $A'B'$.

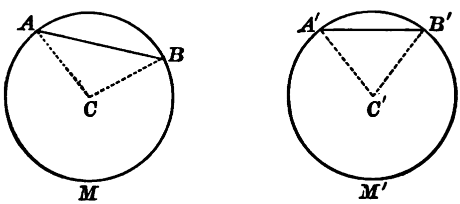
To Prove chord $AB =$ chord $A'B'$.

(Prove $\triangle ABC = \triangle A'B'C'$, by § 63.)

Ex. 2. If two circumferences intersect each other, the distance between their centres is greater than the difference of their radii. (§ 62.)

PROP. VII. THEOREM.

159. *In equal circles, or in the same circle, the greater of two arcs is subtended by the greater chord; each arc being less than a semi-circumference.*



Given, in equal $\circledS$ AMB and $A'M'B'$, arc $AB >$ arc $A'B'$, each arc being $<$ a semi-circumference, and chords AB and $A'B'$.

To Prove chord $AB >$ chord $A'B'$.

Proof. Draw radii AC, BC, $A'C'$, and $B'C'$.

Then in $\triangle$s ABC and $A'B'C'$,

$$AC = A'C' \text{ and } BC = B'C'. \qquad (?)$$

And since, by hyp., arc $AB >$ arc $A'B'$, we have

$$\angle C > \angle C'. \qquad (§ 156, 2)$$

$$\therefore \text{ chord } AB > \text{chord } A'B'. \qquad (§ 91)$$

PROP. VIII. THEOREM.

160. (Converse of Prop. VII.) *In equal circles, or in the same circle, the greater of two chords subtends the greater arc; each arc being less than a semi-circumference.*

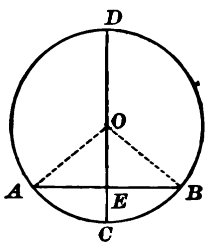
(Fig. of Prop. VII.)

($\angle C > \angle C'$, by § 92; the theorem follows by § 156, 1.)

161. Sch. If each arc is *greater* than a semi-circumference, the greater arc is subtended by the less chord; and conversely the greater chord subtends the less arc.

PROP. IX. THEOREM.

162. *The diameter perpendicular to a chord bisects the chord and its subtended arcs.*



Given, in $\odot ABD$, diameter $CD \perp$ chord AB.

To Prove that CD bisects chord AB, and arcs ACB and ADB.

Proof. Let O be the centre of the $\odot$, and draw radii OA and OB.

Then, $OA = OB$. (?)

Hence, $\triangle OAB$ is isosceles.

Therefore, CD bisects AB, and $\angle AOB$. (§ 94)

Then since $\angle AOC = \angle BOC$, we have

arc $AC =$ arc BC. (§ 154)

Again, $\angle AOD = \angle BOD$. (§ 31, 2)

$\therefore$ arc $AD =$ arc BD. (?)

Hence, CD bisects AB, and arcs ACB and ADB.

163. Cor. *The perpendicular erected at the middle point of a chord passes through the centre of the circle, and bisects the arcs subtended by the chord.*

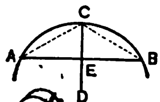
EXERCISES.

3. The diameter which bisects a chord is perpendicular to it and bisects its subtended arcs. (§ 43.)

(Fig. of Prop. IX. Given diameter CD bisecting chord AB.)

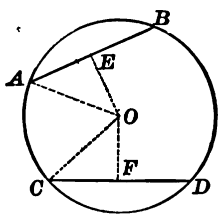
4. The straight line which bisects a chord and its subtended arc is perpendicular to the chord.

(By § 158, chord AC = chord BC.)



PROP. X. THEOREM.

164. *In the same circle, or in equal circles, equal chords are equally distant from the centre.*



Given AB and CD equal chords of $\odot$ ABC, whose centre is O, and lines OE and $OF \perp$ to AB and CD, respectively.

To Prove $OE = OF$. (§ 47)

Proof. Draw radii OA and OC.

Then in rt. $\triangle$ OAE and OCF,

$$OA = OC. \qquad (?)$$

Now, E is the middle point of AB, and F of CD. (§ 162)

$$\therefore AE = CF,$$

being halves of equal chords AB and CD, respectively.

$$\therefore \triangle OAE = \triangle OCF. \qquad (?)$$

$$\therefore OE = OF. \qquad (?)$$

PROP. XI. THEOREM.

165. (Converse of Prop. X.) *In the same circle, or in equal circles, chords equally distant from the centre are equal.*

(Fig. of Prop. X.)

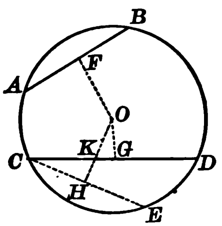
Given O the centre of $\odot ABC$, and AB and CD chords equally distant from O.

To Prove chord $AB =$ chord CD.

(Rt. $\triangle OAE =$ rt. $\triangle OCF$, and $AE = CF$; E is the middle point of AB, and F of CD.)

PROP. XII. THEOREM.

166. *In the same circle, or in equal circles, the less of two chords is at the greater distance from the centre.*



Given, in $\odot ABC$, chord $AB <$ chord CD, and $\perp$s OF and OG drawn from centre O to AB and CD, respectively.

To Prove $OF > OG$.

Proof. Since chord $AB <$ chord CD, we have

arc $AB <$ arc CD. (§ 160)

Lay off arc $CE =$ arc AB, and draw line CE.

$\therefore$ chord $CE =$ chord AB. (§ 158)

Draw line $OH \perp CE$, intersecting CD at K.

$\therefore OH = OF$. (§ 164)

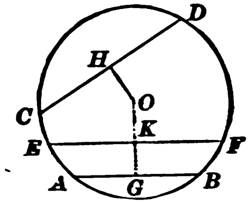
But, $OH > OK$.

And, $OK > OG$. (?)

Whence, OH, or its equal OF, is $> OG$.

PROP. XIII. THEOREM.

167. (Converse of Prop. XII.) ***In the same circle, or in equal circles, if two chords are unequally distant from the centre, the more remote is the less.***



Given O the centre of $\odot ABC$, and chord AB more remote from O than chord CD.

To Prove chord $AB <$ chord CD.

Proof. Draw lines OG and $OH \perp$ to AB and CD respectively, and on OG lay off $OK = OH$.

Through K draw chord $EF \perp OK$.

$\therefore$ chord $EF =$ chord CD. (§ 165)

Now, chord $AB \parallel$ chord EF. (§ 54)

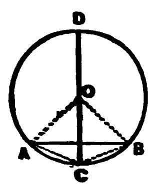
Then it is evident that arc AB is $<$ arc EF, for it is only a portion of arc EF.

$\therefore$ chord $AB <$ chord EF. (§ 159)

$\therefore$ chord $AB <$ chord CD.

168. Cor. ***A diameter of a circle is greater than any other chord;*** for a chord which passes through the centre is greater than any chord which does not. (§ 167)

EXERCISES.

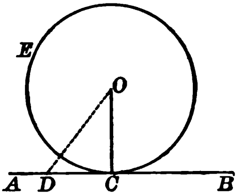


5. The diameter which bisects an arc bisects its chord at right angles.

6. The perpendiculars to the sides of an inscribed quadrilateral at their middle points meet in a common point. (§ 162.)

Prop. XIV. Theorem.

169. *A straight line perpendicular to a radius of a circle at its extremity is tangent to the circle.*



Given line $AB \perp$ to radius OC of $\odot EC$ at C.

To Prove AB tangent to the $\odot$.

Proof. Let D be any point of AB except C, and draw line OD.

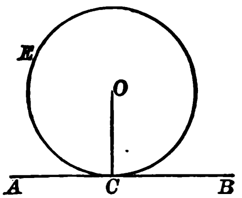
$$\therefore OD > OC. \qquad (?)$$

Therefore, point D lies without the $\odot$.

Then, every point of AB except C lies without the $\odot$, and AB is tangent to the $\odot$. (§ 149)

Prop. XV. Theorem.

170. (Converse of Prop. XIV.) *A tangent to a circle is perpendicular to the radius drawn to the point of contact.*



Given line AB tangent to $\odot EC$ at C, and radius OC.

To Prove $OC \perp AB$.

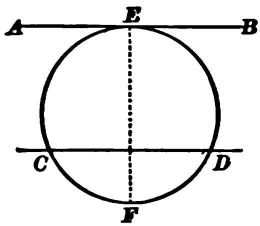
(OC is the shortest line that can be drawn from O to AB.)

171. Cor. *A line perpendicular to a tangent at its point of contact passes through the centre of the circle.*

PROP. XVI. THEOREM.

172. *Two parallels intercept equal arcs on a circumference.*

Case I. *When one line is a tangent and the other a secant.*



Given AB a tangent to ⊙ CED at E, and CD a secant ∥ AB, intersecting the circumference at C and D.

To Prove arc CE = arc DE.

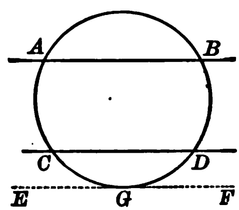
Proof. Draw diameter EF.

$\therefore EF \perp AB.$ (§ 170)

$\therefore EF \perp CD.$ (?)

$\therefore$ arc CE = arc DE. (§ 162)

Case II. *When both lines are secants.*



Given, in ⊙ ABC, AB and CD ∥ secants, intersecting the circumference at A and B, and C and D, respectively.

To Prove arc AC = arc BD.

Proof. Draw tangent $EF \parallel AB$, touching the ⊙ at G.

$\therefore EF \parallel CD.$ (?)

Now, arc AG = arc BG,

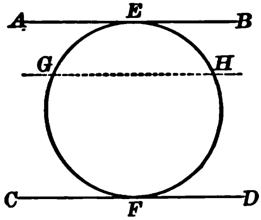
and arc CG = arc DG. (§ 172, Case I)

Subtracting, we have

$$\text{arc } AG - \text{arc } CG = \text{arc } BG - \text{arc } DG.$$

$$\therefore \text{arc } AC = \text{arc } BD.$$

Case III. *When both lines are tangents.*



Given, in $\odot EGF$, AB and $CD \parallel$ tangents, touching the $\odot$ at E and F, respectively.

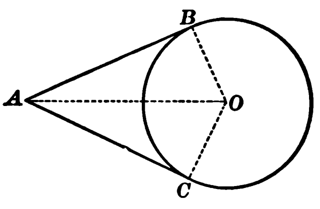
To Prove arc $EGF =$ arc EHF.

(Draw secant $GH \parallel AB$.)

173. Cor. *The straight line joining the points of contact of two parallel tangents is a diameter.*

PROP. XVII. THEOREM.

174. *The tangents to a circle from an outside point are equal.*



(Rt. $\triangle OAB =$ rt. $\triangle OAC$, by § 90; then $AB = AC$.)

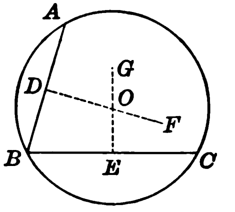
175. Cor. From equal $\triangle$ OAB and OAC,

$$\angle OAB = \angle OAC \text{ and } \angle AOB = \angle AOC.$$

Then, *the line joining the centre of a circle to the point of intersection of two tangents makes equal angles with the tangents, and also with the radii drawn to the points of contact.*

PROP. XVIII. THEOREM.

176. *Through three points, not in the same straight line, a circumference can be drawn, and but one.*



Given points A, B, and C, not in the same straight line.

To Prove that a circumference can be drawn through A, B, and C, and but one.

Proof. Draw lines AB and BC, and lines DF and $EG \perp$ to AB and BC, respectively, at their middle points, meeting at O.

Then O is equally distant from A, B, and C. (§ 137)

Hence, a circumference described with O as a centre and OA as a radius will pass through A, B, and C.

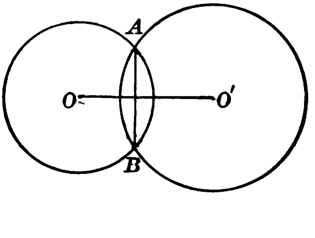
Again, the centre of any circumference drawn through A, B, and C must be in each of the $\perp$s DF and EG. (§ 42)

Then as DF and EG intersect in but one point, only one circumference can be drawn through A, B, and C.

177. Cor. *Two circumferences can intersect in but two points;* for if they had three common points, they would have the same centre, and coincide throughout.

PROP. XIX. THEOREM.

178. *If two circumferences intersect, the straight line joining their centres bisects their common chord at right angles.*



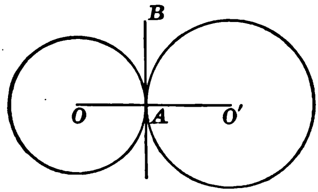
Given O and O' the centres of two ⓢ, whose circumferences intersect at A and B, and lines OO' and AB.

To Prove that OO' bisects AB at rt. $\angle$s.

(The proposition follows by § 43.)

PROP. XX. THEOREM.

179. *If two circles are tangent to each other, the straight line joining their centres passes through their point of contact.*



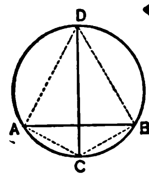
Given O and O' the centres of two ⓢ, which are tangent to line AB at A.

To Prove that str. line joining O and O' passes through A.

(Draw radii OA and $O'A$; since these lines are $\perp$ AB, OAO' is a str. line by § 37; the proposition follows by Ax. 3.)

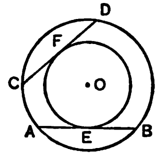
EXERCISES.

7. The straight line which bisects the arcs subtended by a chord bisects the chord at right angles.



8. The tangents to a circle at the extremities of a diameter are parallel.

9. If two circles are concentric, any two chords of the greater which are tangent to the less are equal. (§ 165.)



10. The straight line drawn from the centre of a circle to the point of intersection of two tangents bisects at right angles the chord joining their points of contact. (§ 174.)

ON MEASUREMENT.

180. The *ratio* of a magnitude to another of the same kind is the quotient of the first divided by the second.

Thus, if a and b are quantities of the same kind, the ratio of a to b is $\frac{a}{b}$; it may also be expressed $a : b$.

A magnitude is *measured* by finding its ratio to another magnitude of the same kind, called the *unit of measure.*

The quotient, if it can be obtained exactly as an integer or fraction, is called the *numerical measure* of the magnitude.

181. Two magnitudes of the same kind are said to be *commensurable* when a unit of measure, called a *common measure,* is contained an *integral* number of times in each.

Thus, two lines whose lengths are $2\frac{3}{4}$ and $3\frac{4}{5}$ inches are commensurable; for the common measure $\frac{1}{20}$ inch is contained an integral number of times in each; i.e., 55 times in the first line, and 76 times in the second.

Two magnitudes of the same kind are said to be *incommensurable* when no magnitude of the same kind can be found which is contained an integral number of times in each.

For example, let AB and CD be two lines such that

$$\frac{AB}{CD} = \sqrt{2}.$$

As $\sqrt{2}$ can only be obtained *approximately,* no line, *however small,* can be found which is contained an integral number of times in each line, and AB and CD are incommensurable.

182. A magnitude which is incommensurable with respect to the unit has, strictly speaking, no numerical measure (§ 180); still if CD is the unit of measure, and $\frac{AB}{CD} = \sqrt{2}$, we shall speak of $\sqrt{2}$ as the numerical measure of AB.

183. It is evident from the above that the ratio of two magnitudes of the same kind, whether commensurable or incommensurable, is equal to the ratio of their numerical measures when referred to a common unit.

THE METHOD OF LIMITS.

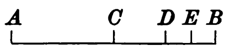
184. A *variable quantity*, or simply a *variable*, is a quantity which may assume, under the conditions imposed upon it, an indefinitely great number of *different* values.

185. A *constant* is a quantity which remains unchanged throughout the same discussion.

186. A *limit* of a variable is a constant quantity, the difference between which and the variable may be made less than any assigned quantity, however small, but cannot be made equal to zero.

In other words, a limit of a variable is a fixed quantity to which the variable approaches indefinitely near, but never actually reaches.

187. Suppose, for example, that a point moves from A towards B under the condition that it shall move, during successive equal intervals of time, first from A to C, half-way between A and B; then to D, half-way between C and B; then to E, half-way between D and B; and so on indefinitely.



In this case, the distance between the moving point and B can be made less than any assigned distance, however small, but cannot be made equal to 0.

Hence, the distance from A to the moving point is a variable which approaches the constant distance AB as a limit.

Again, the distance from the moving point to B is a variable which approaches the limit 0.

As another illustration, consider the series

$$1, \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \tfrac{1}{16}, \cdots,$$

where each term after the first is one-half the preceding.

In this case, by taking terms enough, the last term may be made less than any assigned number, however small, but cannot be made actually equal to 0.

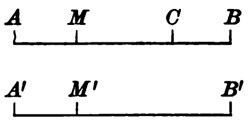
Then, the last term of the series is a variable which approaches the limit 0 when the number of terms is indefinitely increased.

Again, the sum of the first two terms is $1\frac{1}{2}$;
the sum of the first three terms is $1\frac{3}{4}$;
the sum of the first four terms is $1\frac{7}{8}$; etc.

In this case, by taking terms enough, the sum of the terms may be made to differ from 2 by less than any assigned number, however small, but cannot be made actually equal to 2.

Then, the sum of the terms of the series is a variable which approaches the limit 2 when the number of terms is indefinitely increased.

188. The Theorem of Limits. *If two variables are always equal, and each approaches a limit, the limits are equal.*



Given AM and $A'M'$ two variables, which are always equal, and approach the limits AB and $A'B'$, respectively.

To Prove $AB = A'B'$.

Proof. If possible, let AB be $> A'B'$; and lay off, on AB, $AC = A'B'$.

Then, variable AM may have values $> AC$, while variable $A'M'$ is restricted to values $< AC$; which is contrary to the hypothesis that the variables are always equal.

Hence, AB cannot be $> A'B'$.

In like manner, it may be proved that AB cannot be $< A'B'$.

Therefore, since AB can be neither $>$, nor $< A'B'$, we have

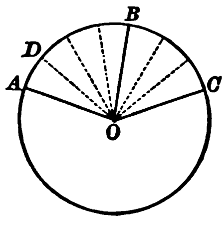
$$AB = A'B'.$$

MEASUREMENT OF ANGLES.

PROP. XXI. THEOREM.

189. *In the same circle, or in equal circles, two central angles are in the same ratio as their intercepted arcs.*

Case I. *When the arcs are commensurable* (§ 181).



Given, in $\odot ABC$, AOB and BOC central $\angle$s intercepting commensurable arcs AB and BC, respectively.

To Prove $$\frac{\angle AOB}{\angle BOC} = \frac{\text{arc } AB}{\text{arc } BC}.$$

Proof. Since, by hyp., arcs AB and BC are commensurable, let arc AD be a common measure of arcs AB and BC; and suppose it to be contained 4 times in arc AB, and 3 times in arc BC.

$$\therefore \frac{\text{arc } AB}{\text{arc } BC} = \frac{4}{3}. \qquad (1)$$

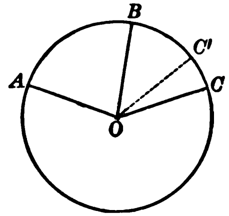
Drawing radii to the several points of division of arc AC, $\angle AOB$ will be divided into 4 $\angle$s, and $\angle BOC$ into 3 $\angle$s, all of which $\angle$s are equal. (§ 155)

$$\therefore \frac{\angle AOB}{\angle BOC} = \frac{4}{3}. \qquad (2)$$

From (1) and (2), we have

$$\frac{\angle AOB}{\angle BOC} = \frac{\text{arc } AB}{\text{arc } BC}. \qquad (?)$$

Case II. *When the arcs are incommensurable* (§ 181).



Given, in $\odot ABC, AOB$ and BOC central $\angle$s intercepting incommensurable arcs AB and BC, respectively.

To Prove $$\frac{\angle AOB}{\angle BOC} = \frac{\text{arc } AB}{\text{arc } BC}.$$

Proof. Let arc AB be divided into any number of equal arcs, and let one of these arcs be applied to arc BC as a unit of measure.

Since arcs AB and BC are incommensurable, a certain number of the equal arcs will extend from B to C', leaving a remainder $C'C$ less than one of the equal arcs.

Draw radius OC'.

Then, since by const., arcs AB and BC' are commensurable,

$$\frac{\angle AOB}{\angle BOC'} = \frac{\text{arc } AB}{\text{arc } BC'}. \qquad \text{(§ 189, Case I.)}$$

Now let the number of subdivisions of arc AB be indefinitely increased.

Then the unit of measure will be indefinitely diminished; and the remainder $C'C$, being always less than the unit, will approach the limit 0.

Then $\angle BOC'$ will approach the limit $\angle BOC$,

and arc BC' will approach the limit arc BC.

Hence, $\dfrac{\angle AOB}{\angle BOC'}$ will approach the limit $\dfrac{\angle AOB}{\angle BOC}$,

and $\dfrac{\text{arc } AB}{\text{arc } BC'}$ will approach the limit $\dfrac{\text{arc } AB}{\text{arc } BC}$.

Now, $\frac{\angle AOB}{\angle BOC'}$ and $\frac{\text{arc } AB}{\text{arc } BC'}$ are variables which are always equal, and approach the limits $\frac{\angle AOB}{\angle BOC}$ and $\frac{\text{arc } AB}{\text{arc } BC}$, respectively.

By the Theorem of Limits, these limits are equal. (§ 188)

$$\therefore \frac{\angle AOB}{\angle BOC} = \frac{\text{arc } AB}{\text{arc } BC}.$$

190. Sch. The usual unit of measure for arcs is the *degree*, which is the ninetieth part of a quadrant (§ 146).

The degree of arc is divided into sixty equal parts, called *minutes*, and the minute into sixty equal parts, called *seconds*.

If the sum of two arcs is a quadrant, or 90°, one is called the *complement* of the other; if their sum is a semi-circumference, or 180°, one is called the *supplement* of the other.

191. Cor. I. By § 154, equal central ⦞s, in the same ⊙s, intercept equal arcs on the circumference.

Hence, if the angular magnitude about the centre of a ⊙ be divided into four equal ⦞s, each ∠ will intercept an arc equal to one-fourth of the circumference.

That is, *a right central angle intercepts a quadrant on the circumference.* (§ 35)

192. Cor. II. By § 189, a central ∠ of n degrees bears the same ratio to a rt. central ∠ that its intercepted arc bears to a quadrant.

But a central ∠ of n degrees is $\frac{n}{90}$ of a rt. central ∠.

Hence, its intercepted arc is $\frac{n}{90}$ of a quadrant, or an arc of n degrees.

The above principle is usually expressed as follows:

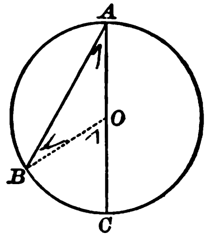
A central angle is measured by its intercepted arc.

This means simply that the number of angular degrees in a central angle is equal to the number of degrees of arc in its intercepted arc.

PROP. XXII. THEOREM.

193. *An inscribed angle is measured by one-half its intercepted arc.*

Case I. *When one side of the angle is a diameter.*



Given AC a diameter, and AB a chord, of $\odot ABC$.

To Prove that $\angle BAC$ is measured by $\frac{1}{2}$ arc BC.

Proof. Draw radius OB; then, $OA = OB$. (?)

Then $\triangle OAB$ is isosceles, and $\angle B = \angle A$. (?)

But since BOC is an ext. $\angle$ of $\triangle OAB$,

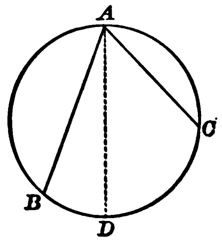
$$\angle BOC = \angle A + \angle B. \qquad (\S\ 85,\ 1)$$

$$\therefore \angle BOC = 2\angle A, \text{ or } \angle A = \tfrac{1}{2}\angle BOC.$$

But, $\angle BOC$ is measured by arc BC. (§ 192)

Whence, $\angle A$ is measured by $\frac{1}{2}$ arc BC.

Case II. *When the centre is within the angle.*



Given AB and AC chords of $\odot ABC$, and the centre of the $\odot$ within $\angle BAC$.

To Prove that $\angle BAC$ is measured by $\frac{1}{2}$ arc BC.

Proof. Draw diameter AD.

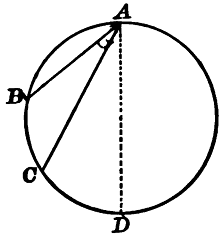
Then, $\angle BAD$ is measured by $\frac{1}{2}$ arc BD,

and $\angle CAD$ is measured by $\frac{1}{2}$ arc CD. (§ 193, Case I)

$\therefore \angle BAD + \angle CAD$ is measured by $\frac{1}{2}$ arc $BD + \frac{1}{2}$ arc CD.

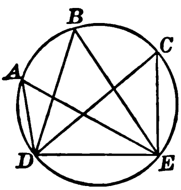
$\therefore \angle BAC$ is measured by $\frac{1}{2}$ arc BC.

Case III. *When the centre is without the angle.*



(The proof is left to the pupil.)

194. Cor. I. *Angles inscribed in the same segment are equal.*

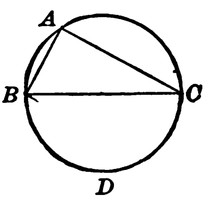


Given A, B, and C ∠s inscribed in segment ADE of ⊙ ABC.

To Prove $\angle A = \angle B = \angle C$.

(The proposition follows by § 193.)

195. Cor. II. *An angle inscribed in a semicircle is a right angle.*

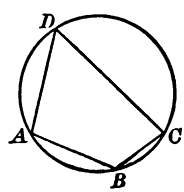


Given BC a diameter, and AB and AC chords, of ⊙ ABD.

To Prove $\angle BAC$ a rt. ∠.

Proof. $\angle BAC$ is measured by $\frac{1}{2}$ of 180°, or 90°. (§ 193)

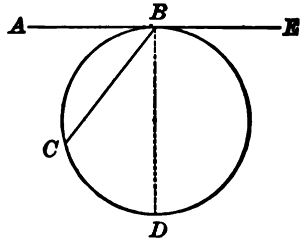
196. Cor. III. *The opposite angles of an inscribed quadrilateral are supplementary.*



For their sum is measured by $\frac{1}{2}$ of 360°, or 180°. (?)

Prop. XXIII. Theorem.

197. *The angle between a tangent and a chord is measured by one-half its intercepted arc.*



Given AE a tangent to $\odot BCD$ at B, and BC a chord.

To Prove that $\angle ABC$ is measured by $\frac{1}{2}$ arc BC.

Proof. Draw diameter BD; then, $BD \perp AE$. (?)

Now a rt. $\angle$ is measured by one-half a semi-circumference.

$\therefore \angle ABD$ is measured by $\frac{1}{2}$ arc BCD.

Also, $\angle CBD$ is measured by $\frac{1}{2}$ arc CD. (§ 193)

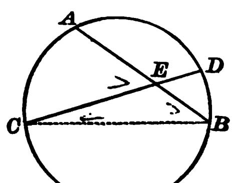
$\therefore \angle ABD - \angle CBD$ is measured by $\frac{1}{2}$ arc $BCD - \frac{1}{2}$ arc CD.

$\therefore \angle ABC$ is measured by $\frac{1}{2}$ arc BC.

Similarly, $\angle EBC$ is measured by $\frac{1}{2}$ arc BDC.

Prop. XXIV. Theorem.

198. *The angle between two chords, intersecting within the circumference, is measured by one-half the sum of its intercepted arc, and the arc intercepted by its vertical angle.*



Given, in $\odot ABC$, chords AB and CD intersecting within the circumference at E.

To Prove that

$\angle AEC$ is measured by $\frac{1}{2}$ (arc AC + arc BD).

Proof. Draw chord BC.

Then, since AEC is an ext. $\angle$ of $\triangle BCE$,

$$\angle AEC = \angle B + \angle C. \qquad (?)$$

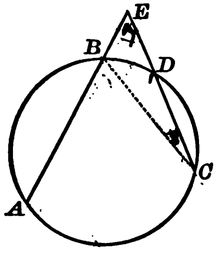
But, $\angle B$ is measured by $\frac{1}{2}$ arc AC,

and $\angle C$ is measured by $\frac{1}{2}$ arc BD. (?)

$\therefore \angle AEC$ is measured by $\frac{1}{2}$ (arc AC + arc BD).

PROP. XXV. THEOREM.

199. *The angle between two secants, intersecting without the circumference, is measured by one-half the difference of the intercepted arcs.*



Given, in $\odot ABC$, secants AE and CE intersecting without the circumference at E, and intersecting the circumference at A and B, and C and D, respectively.

To Prove that $\angle E$ is measured by $\frac{1}{2}$ (arc AC − arc BD).

Proof. Draw chord BC.

Then since ABC is an ext. $\angle$ of $\triangle BCE$,

$$\angle ABC = \angle E + \angle C. \qquad (?)$$

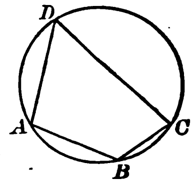
$$\therefore \angle E = \angle ABC - \angle C.$$

But, $\angle ABC$ is measured by $\frac{1}{2}$ arc AC,

and $\angle C$ is measured by $\frac{1}{2}$ arc BD. (?)

$\therefore \angle E$ is measured by $\frac{1}{2}$ (arc AC − arc BD).

200. Cor. (Converse of § 196.) *If the opposite angles of a quadrilateral are supplementary, the quadrilateral can be inscribed in a circle.*



Given, in quadrilateral $ABCD$, $\angle A$ sup. to $\angle C$, and $\angle B$ to $\angle D$; also, a circumference drawn through A, B, and C. (§ 176)

To Prove that D lies on the circumference.

Proof. Since $\angle D$ is sup. to $\angle B$, it is measured by $\frac{1}{2}$ arc ABC. (§ 193)

Then, D must lie on the circumference; for if it were within the $\odot$, $\angle D$ would be measured by $\frac{1}{2}$ an arc $> ABC$; and if it were without the $\odot$, $\angle D$ would be measured by $\frac{1}{2}$ an arc $< ABC$. (§§ 198, 199)

PROP. XXVI. THEOREM.

201. *The angle between a secant and a tangent, or two tangents, is measured by one-half the difference of the intercepted arcs.*

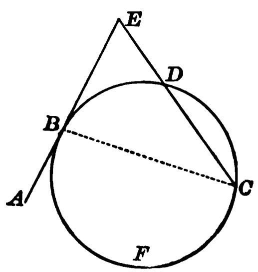


Fig. 1.

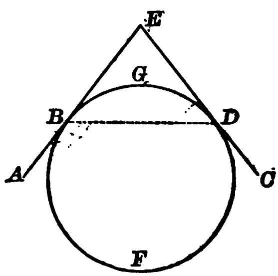


Fig. 2.

1. **Given** AE a tangent to $\odot$ BDC at B, and EC a **secant** intersecting the circumference at C and D. (Fig. 1.)

To Prove that $\angle E$ is measured by $\frac{1}{2}$ (arc BFC — arc BD).
(We have $\angle E = \angle ABC - \angle C$.)

2. (In Fig. 2, $\angle E = \angle ABD - \angle BDE$; then use § 197.)

202. Cor. Since a circumference is an arc of 360°, we have

$$\begin{aligned}\tfrac{1}{2}(\text{arc } BFD - \text{arc } BGD) &= \tfrac{1}{2}(360° - \text{arc } BGD - \text{arc } BGD)\\ &= \tfrac{1}{2}(360° - 2 \text{ arc } BGD)\\ &= 180° - \text{arc } BGD.\end{aligned}$$

Then, $\angle E$ is measured by $180° - \text{arc } BGD$.

Hence, *the angle between two tangents is measured by the supplement of the smaller of the two intercepted arcs.*

EXERCISES.

11. If, in figure of § 197, arc $BC = 107°$, how many degrees are there in angles ABC and EBC?

12. If, in figure of § 198, arc $AC = 74°$, and $\angle AEC = 51°$, how many degrees are there in arc BD?

13. If, in figure of § 199, arc $AC = 117°$, and $\angle C = 14°$, how many degrees are there in angle E?

14. If, in figure of § 199, AC is a quadrant, and $\angle E = 39°$, how many degrees are there in arc BD?

15. If, in Fig. 1 of § 201, arc $BFC = 197°$, and arc $CD = 75°$, how many degrees are there in angle E?

16. If, in Fig. 1 of § 201, $\angle E = 53°$, and arc BD is one-fifth of the circumference, how many degrees are there in arc BFC?

17. If, in Fig. 2 of § 201, arc BFD is thirteen-sixteenths of the circumference, how many degrees are there in angle E?

18. Three consecutive sides of an inscribed quadrilateral subtend arcs of 82°, 99°, and 67° respectively. Find each angle of the quadrilateral in degrees, and the angle between its diagonals.

19. Prove Prop. XXIV. by drawing through B a chord parallel to CD. (§ 172.)

20. Prove Prop. XXV. by drawing through B a chord parallel to CD.

21. Prove Prop. XXVI. for Fig. 1 by drawing through D a chord parallel to AE.

22. An angle inscribed in a segment greater than a semicircle is acute; and an angle inscribed in a segment less than a semicircle is obtuse. (§ 193.)

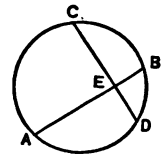
23. In an inscribed trapezoid the non-parallel sides are equal, and also the diagonals.

(The non-parallel sides, and also the diagonals, subtend equal arcs.)

24. If the inscribed and circumscribed circles of a triangle are concentric, prove the triangle equilateral. (§ 165.)

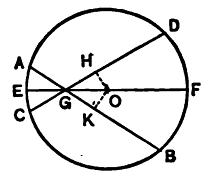
25. If AB and AC are the tangents from point A to the circle whose centre is O, prove $\angle BAC = 2 \angle OBC$. (Ex. 10, p. 87.)

26. If two chords intersect at right angles within the circumference of a circle, the sum of the opposite intercepted arcs is equal to a semi-circumference.

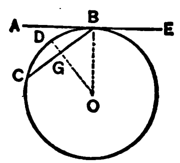


27. Two intersecting chords which make equal angles with the diameter passing through their point of intersection are equal. (§ 165.)

(Prove that $OH = OK$.)



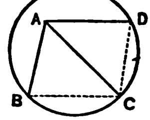
28. Prove Prop. XXIII. by drawing a radius perpendicular to BC. (§ 162.)



29. If AB and AC are two chords of a circle making equal angles with the tangent at A, prove $AB = AC$.

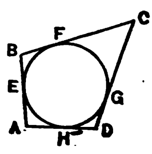
30. From a given point within a circle and not coincident with the centre, not more than two equal straight lines can be drawn to the circumference.

(If possible, let AB, AC, and AD be three equal straight lines from point A to circumference BCD; then, by § 163, A must coincide with the centre.)



31. The sum of two opposite sides of a circumscribed quadrilateral is equal to the sum of the other two sides. (§ 174.)

(To prove $AB + CD = AD + BC$.)



32. Prove Prop. VI. by superposition.

33. In a circumscribed trapezoid, the straight line joining the middle points of the non-parallel sides is equal to one-fourth the perimeter of the trapezoid. (§ 132.)

34. If the opposite sides of a circumscribed quadrilateral are parallel, the figure is a rhombus or a square. (Ex. 31.)

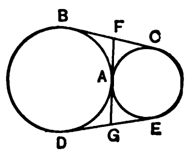
(Prove the sides all equal.)

35. If tangents be drawn to a circle at the extremities of any pair of diameters which are not perpendicular to each other, the figure formed is a rhombus. (Ex. 34.)

36. If the angles of a circumscribed quadrilateral are right angles, the figure is a square.

37. If two circles are tangent to each other at point A, the tangents to them from any point in the common tangent which passes through A are equal. (§ 174.)

38. If two circles are tangent to each other externally at point A, the common tangent which passes through A bisects the other two common tangents. (Ex. 37.)



(To prove that FG bisects BC and DE.)

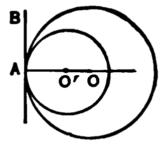
39. The bisector of the angle between two tangents to a circle passes through the centre.

(The bisector of the $\angle$ between the tangents bisects at rt. $\angle s$ the chord joining their points of contact.)

40. The bisectors of the angles of a circumscribed quadrilateral pass through a common point.

41. If AB is one of the non-parallel sides of a trapezoid circumscribed about a circle whose centre is O, prove AOB a right angle. (§ 175.)

42. If two circles are tangent to each other internally, the distance between their centres is equal to the difference of their radii.



43. Prove the theorem of § 168 by drawing radii to the extremities of the chord. (Ax. 4.)

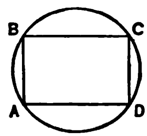
44. Prove the theorem of § 202 by drawing radii to the points of contact of the tangents. (§ 192.)

45. If any number of angles are inscribed in the same segment, their bisectors pass through a common point. (§ 193.)

46. Prove Prop. XIII. by *Reductio ad Absurdum*. (§§ 164, 166.)

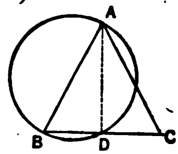
47. Two chords perpendicular to a third chord at its extremities are equal. (§ 158.)

48. If two opposite sides of an inscribed quadrilateral are equal and parallel, the figure is a rectangle.
(Arc BCD is a semi-circumference.)

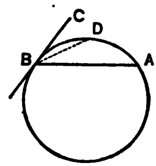


49. If the diagonals of an inscribed quadrilateral intersect at the centre of the circle, the figure is a rectangle. (§ 195.)

50. The circle described on one of the equal sides of an isosceles triangle as a diameter, bisects the base. (§ 195.)



51. If a tangent be drawn to a circle at the extremity of a chord, the middle point of the subtended arc is equally distant from the chord and from the tangent.
(BD bisects $\angle ABC$.)



52. If sides AB, BC, and CD of an inscribed quadrilateral subtend arcs of 99°, 106°, and 78°, respectively, and sides BA and CD produced meet at E, and sides AD and BC at F, find the number of degrees in angles AED and AFB.

53. If O is the centre of the circumscribed circle of triangle ABC, and OD be drawn perpendicular to BC, prove

$$\angle BOD = \angle A. \quad (\S\ 192.)$$

54. If D, E, and F are the points of contact of sides AB, BC, and CA respectively of a triangle circumscribed about a circle, prove

$$\angle DEF = 90° - \tfrac{1}{2} A. \quad (\S\ 202.)$$

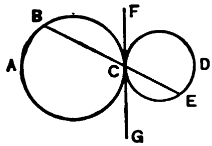
55. If sides AB and BC of an inscribed quadrilateral $ABCD$ subtend arcs of 69° and 112°, respectively, and angle AED between the diagonals is 87°, how many degrees are there in each angle of the quadrilateral?

56. If any number of parallel chords be drawn in a circle, their middle points lie in the same straight line. (§ 162.)

57. What is the locus of the middle points of a system of parallel chords in a circle?

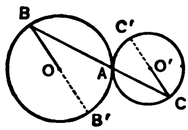
58. What is the locus of the middle points of a system of chords of given length in a circle?

59. If two circles are tangent to each other, any straight line drawn through their point of contact subtends arcs of the same number of degrees on their circumferences. (§ 197.)



(Let the pupil draw the figure for the case when the Ⓢ are tangent internally.)

60. If a straight line be drawn through the point of contact of two circles which are tangent to each other externally, terminating in their circumferences, the radii drawn to its extremities are parallel. (§ 73.)



(Let the pupil state the corresponding theorem for the case when the Ⓢ are tangent internally.)

61. If a straight line be drawn through the point of contact of two circles which are tangent to each other externally, terminating in their circumferences, the tangents at its extremities are parallel. (§ 73.)

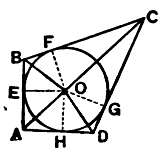
(Let the pupil state the corresponding theorem for the case when the Ⓢ are tangent internally.)

62. If sides AB and DC of inscribed quadrilateral $ABCD$ be produced to meet at E, prove that triangles ACE and BDE, and also triangles ADE and BCE, are mutually equiangular.

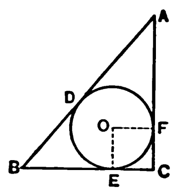
(For second part, see § 196.)

63. The sum of the angles subtended at the centre of a circle by two opposite sides of a circumscribed quadrilateral is equal to two right angles. (§ 175.)

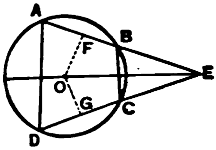
(To prove $\angle AOB + \angle COD = 180°$.)



64. If a circle is inscribed in a right triangle, the sum of its diameter and the hypotenuse is equal to the sum of the legs. (§ 174.)

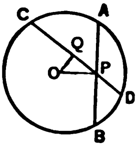


65. If a circle be described on the radius of another circle as a diameter, any chord of the greater passing through the point of contact of the circles is bisected by the circumference of the smaller. (§ 195.)



66. If sides AB and CD of inscribed quadrilateral $ABCD$ make equal angles with the diameter passing through their point of intersection, prove $AB = CD$. (§ 165.)

67. If angles A, B, and C of circumscribed quadrilateral $ABCD$ are 128°, 67°, and 112°, respectively, and sides AB, BC, CD, and DA are tangent to the circle at points E, F, G, and H, respectively, find the number of degrees in each angle of quadrilateral $EFGH$.



68. The chord drawn through a given point within a circle, perpendicular to the diameter passing through the point, is the least chord which can be drawn through the given point. (§ 165.)

(Given chords AB and CD drawn through P, and $AB \perp OP$. To prove $AB < CD$.)

69. If ADB, BEC, and CFA are angles inscribed in segments ABD, BCE, and ACF, respectively, exterior to inscribed triangle ABC, prove their sum equal to four right angles. (§ 196.)

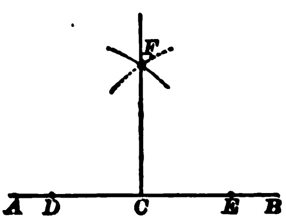
Note. For additional exercises on Book II., see p. 222.

CONSTRUCTIONS.

PROP. XXVII. PROBLEM.

203. *At a given point in a straight line to erect a perpendicular to that line.*

First Method.



Given C any point in line AB.

Required to draw a line $\perp$ to AB at C.

Construction. Take points D and E on AB equally distant from C.

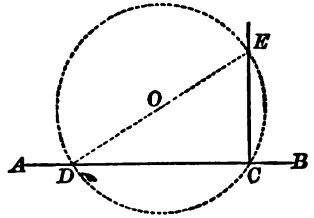
With D and E as centres, and with equal radii, describe arcs intersecting at F, and draw line CF.

Then, CF is $\perp$ to AB at C.

Proof. By cons., C and F are each equally distant from D and E.

Whence, CF is $\perp$ to DE at its middle point. (?)

Second Method.



Given C any point in line AB.

Required to draw a line $\perp$ to AB at C.

Construction. With any point O without line AB as a centre, and distance OC as a radius, describe a circumference intersecting AB at C and D.

Draw diameter DE, and line CE.

Then, CE is $\perp$ to AB at C.

Proof. $\angle DCE$, being inscribed in a semicircle, is a rt. $\angle$. (§ 195)

$$\therefore CE \perp CD.$$

Note. The second method of construction is preferable when the given point is near the end of the line.

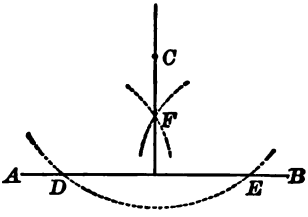
EXERCISES.

70. Given the base and altitude of an isosceles triangle, to construct the triangle.

71. Given an acute angle, to construct its complement.

Prop. XXVIII. Problem.

204. *From a given point without a straight line to draw a perpendicular to that line.*



Given C any point without line AB.

Required to draw from C a line $\perp$ to AB.

Construction. With C as a centre, and any convenient radius, describe an arc intersecting AB at D and E.

With D and E as centres, and with equal radii, describe arcs intersecting at F.

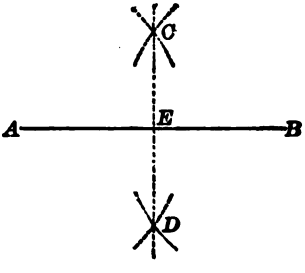
Draw line CF.

Then, $CF \perp AB$.

Proof. Since, by cons., C and F are each equally distant from D and E, CF is $\perp$ to DE at its middle point. (?)

Prop. XXIX. Problem.

205. *To bisect a given straight line.*



Given line AB.

Required to bisect AB.

Construction. With A and B as centres, and with equal radii, describe arcs intersecting at C and D.

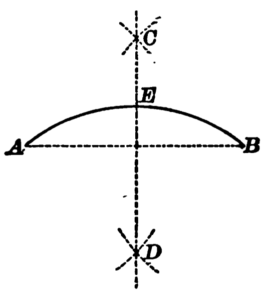
Draw line CD intersecting AB at E.

Then, E is the middle point of AB.

(The proof is left to the pupil.)

Prop. XXX. Problem.

206. *To bisect a given arc.*



Given arc AB.

Required to bisect arc AB.

Construction. With A and B as centres, and with equal radii, describe arcs intersecting at C and D.

Draw line CD intersecting arc AB at E.

Then E is the middle point of arc AB.

Proof. Draw chord AB.

Then, CD is $\perp$ to chord AB at its middle point. (?)

Whence, CD bisects arc AB. (§ 163)

EXERCISES.

72. Given an angle, to construct its supplement.

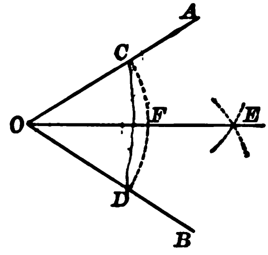
73. Given a side of an equilateral triangle, to construct the triangle.

74. To construct an angle of 60° (Ex. 73); of 30° (Ex. 71).

75. To construct an angle of 120° (Ex. 72); of 150°.

Prop. XXXI. Problem.

207. *To bisect a given angle.*



Given $\angle AOB$.

Required to bisect $\angle AOB$.

Construction. With O as a centre, and any convenient radius, describe an arc intersecting OA at C, and OB at D.

With C and D as centres, and with the same radius as before, describe arcs intersecting at E, and draw line OE.

Then, OE bisects $\angle AOB$.

Proof. Let OE intersect arc CD at F.

By cons., O and E are each equally distant from C and D.

Whence, OE bisects arc CD at F (§ 206).

$$\therefore \angle COF = \angle DOF. \qquad (?)$$

That is, OE bisects $\angle AOB$.

Prop. XXXII. Problem.

208. *With a given vertex and a given side, to construct an angle equal to a given angle.*

Given O **the vertex, and** OA **a side, of an** $\angle$, **and** $\angle O'$.

Required to construct, with O as the vertex and OA as a side, an $\angle$ equal to $\angle O'$.

Construction. With O' as a centre, and any convenient radius, describe an arc intersecting the sides of $\angle O'$ at C and D; and draw chord CD.

With O as a centre, and with the same radius as before, describe the indefinite arc AE.

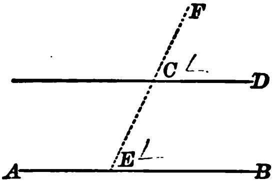
With A as a centre and CD as a radius, describe an arc intersecting arc AE at B, and draw line OB.

Then, $$\angle AOB = \angle O'.$$

(The chords of arcs AB and CD are equal, and the proposition follows by §§ 157 and 155.)

Prop. XXXIII. Problem.

209. *Through a given point without a given straight line, to draw a parallel to the line.*



Given C any point without line AB.

Required to draw through C a line $\parallel$ to AB.

Construction. Through C draw any line EF, meeting AB at E, and construct $\angle FCD = \angle CEB$. (§ 208)

Then, $$CD \parallel AB. \quad (?)$$

EXERCISES.

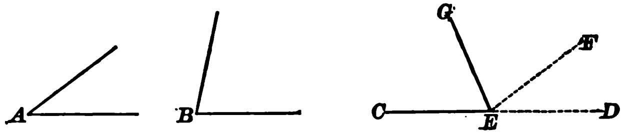
76. To construct an angle of 45°; of 135°; of $22\frac{1}{2}$°; of $67\frac{1}{2}$°.

77. Through a given point without a straight line to draw a line making a given angle with that line.

(Draw through the given point a $\parallel$ to the given line.)

PROP. XXXIV. PROBLEM.

210. *Given two angles of a triangle, to find the third.*



Given A and B two ∠s of a △.

Required to construct the third ∠.

Construction. At any point E of the indefinite line CD, construct $\angle DEF = \angle A$. (§ 208)

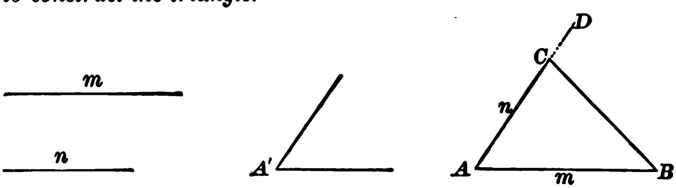
Also, $\angle FEG$ adjacent to $\angle DEF$, and equal to $\angle B$.

Then, $\angle CEG$ is the required ∠.

(The proof is left to the pupil.)

PROP. XXXV. PROBLEM.

211. *Given two sides and the included angle of a triangle, to construct the triangle.*



Given m and n two sides of a △, and A' their included ∠.

Required to construct the △.

Construction. Draw line $AB = m$.

Construct $\angle BAD = \angle A'$. (§ 208)

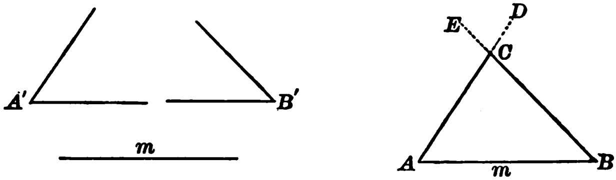
On AD take $AC = n$, and draw line BC.

Then, ABC is the required △.

212. Sch. The problem is possible for any values of the given parts.

PROP. XXXVI. PROBLEM.

213. *Given a side and two adjacent angles of a triangle, to construct the triangle.*



Given a side m, and the adj. ∡s A' and B' of a △.

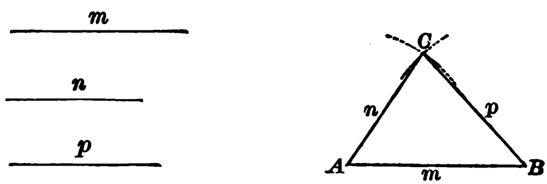
(The construction is left to the pupil.)

214. Sch. I. If a side and *any* two angles of a triangle are given, the third angle may be found by § 210, and the triangle may then be constructed as in § 213.

215. Sch. II. The problem is impossible when the sum of the given angles is equal to, or greater than, two right angles. (§ 84)

PROP. XXXVII. PROBLEM.

216. *Given the three sides of a triangle, to construct the triangle.*



Given m, n, and p the three sides of a △.

Required to construct the △.

Construction. Draw line $AB = m$.

With A as a centre and n as a radius, describe an arc; with B as a centre and p as a radius, describe an arc intersecting the former arc at C, and draw lines AC and BC.

Then, ABC is the required △.

217. Sch. The problem is impossible when one of the given sides is equal to, or greater than, the sum of the other two. (§ 61)

Prop. XXXVIII. Problem.

218. *Given two sides of a triangle, and the angle opposite to one of them, to construct the triangle.*

Given m and n two sides of a $\triangle$, and A' the $\angle$ opposite to n.

Required to construct the $\triangle$.

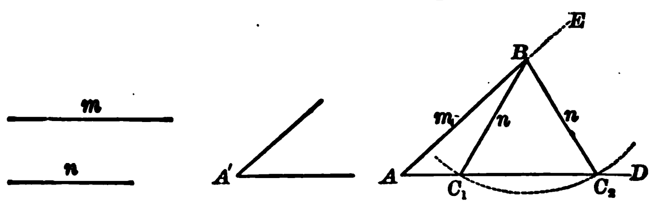
Construction. Construct $\angle DAE = \angle A'$ (§ 208), and on AE take $AB = m$.

With B as a centre and n as a radius, describe an arc.

Case I. *When A' is acute, and $m > n$.*

There may be three cases:

1. The arc may intersect AD in two points.



Let C_1 and C_2 be the points in which the arc intersects AD, and draw lines BC_1 and BC_2.

Then, either ABC_1 or ABC_2 is the required $\triangle$.

Note. This is called the *ambiguous case.*

2. The arc may be tangent to AD.

In this case there is but one $\triangle$.

And since a tangent to a $\odot$ is $\perp$ to the radius drawn to the point of contact (§ 170), the $\triangle$ is a *right* $\triangle$.

3. The arc may not intersect AD at all.

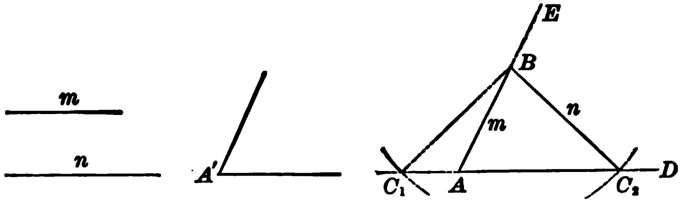
In this case the problem is impossible.

Case II. *When A' is acute, and $m = n$.*

In this case, the arc intersects AD in two points, one of which is A.

Then there is but one $\triangle$; an *isosceles* $\triangle$.

Case III. *When A' is acute, and $m < n$.*



In this case, the arc intersects AD in two points.

Let C_1 and C_2 be the points in which the arc intersects AD, and draw lines BC_1 and BC_2.

Now $\triangle ABC_1$ does not satisfy the conditions of the problem, since it does not contain the given $\angle A'$.

Then there is but one $\triangle$; $\triangle ABC_2$.

Case IV. *When A' is right or obtuse, and $m < n$.*

In each of these cases, the arc intersects AD in two points on opposite sides of A.

Then there is but one $\triangle$.

219. Sch. If A' is right or obtuse, and $m = n$ or $m > n$, the problem is impossible; for the side opposite the right or obtuse angle in a triangle must be the greatest side of the triangle. (§ 99)

The pupil should construct the triangle corresponding to each case of § 218.

EXERCISES.

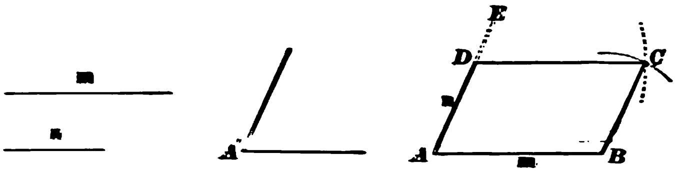
78. Given one of the equal sides and the altitude of an isosceles triangle, to construct the triangle.

What restriction is there on the values of the given lines?

79. Given two diagonals of a parallelogram and their included angle, to construct the parallelogram. (§ 111.)

PROP. XXXIX. PROBLEM.

220. *Given two sides and the included angle of a parallelogram, to construct the parallelogram.*

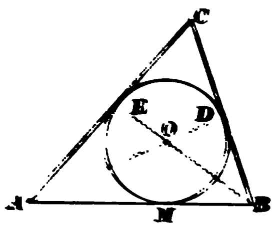


Given m and n two sides, and A' the included $\angle$, of a $\square$.

(The construction and proof are left to the pupil.)

PROP. XL. PROBLEM.

221. *To inscribe a circle in a given triangle.*



Given $\triangle ABC$.

Required to inscribe a $\odot$ in $\triangle ABC$.

Construction. Draw lines AD and BE bisecting $\angle$s A and B respectively (§ 207).

From their intersection O, draw line $OM \perp AB$ (§ 204).

With O as a centre and OM as a radius, describe a $\odot$.

This $\odot$ will be tangent to AB, BC, and CA.

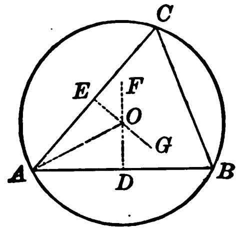
(The proof is left to the pupil: see § 135.)

Ex. 89. To construct a right triangle, having given the hypotenuse and an acute angle.

(The other acute $\angle$ is the complement of the given $\angle$.)

PROP. XLI. PROBLEM.

222. *To circumscribe a circle about a given triangle.*



Given $\triangle ABC$.

Required to circumscribe a $\odot$ about $\triangle ABC$.

Construction. Draw lines DF and $EG \perp$ to AB and AC, respectively, at their middle points (§ 205).

Let DF and EG intersect at O.

With O as a centre, and OA as a radius, describe a $\odot$.

The circumference will pass through A, B, and C.

(The proof is left to the pupil; see § 137.)

223. Sch. The above construction serves to describe a circumference through three given points not in the same straight line, or to find the centre of a given circumference or arc.

EXERCISES.

81. To construct a right triangle, having given a leg and the opposite acute angle.

(Construct the complement of the given $\angle$.)

82. Given the base and the vertical angle of an isosceles triangle, to construct the triangle.

(Each of the equal $\angle$s is the complement of one-half the vertical $\angle$.)

83. Given the altitude and one of the equal angles of an isosceles triangle, to construct the triangle.

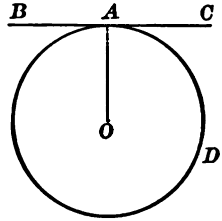
(One-half the vertical $\angle$ is the complement of each of the equal $\angle$s.)

84. To circumscribe a circle about a given rectangle.

(Draw the diagonals.)

PROP. XLII. PROBLEM.

224. *To draw a tangent to a circle through a given point on the circumference.*



Given A any point on the circumference of $\odot AD$.

Required to draw through A a tangent to $\odot AD$.

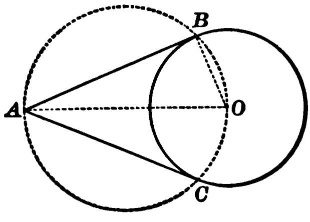
Construction. Draw radius OA.

Through A draw line $BC \perp OA$ (§ 203).

Then, BC will be tangent to $\odot AD$. (?)

PROP. XLIII. PROBLEM.

225. *To draw a tangent to a circle through a given point without the circle.*



Given A any point without $\odot BC$.

Required to draw through A a tangent to $\odot BC$.

Construction. Let O be the centre of $\odot BC$, and draw line OA.

On OA as a diameter, describe a circumference, cutting the given circumference at B and C.

Draw lines AB and AC.

Then, AB and AC are tangents to $\odot$ BC.

Proof. Draw line OB.

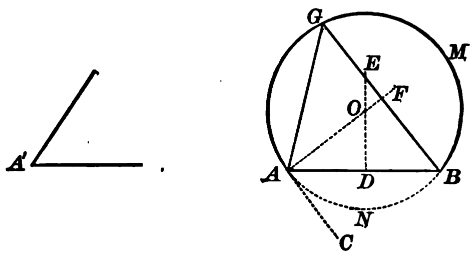
$\angle$ ABO, being inscribed in a semicircle, is a rt. $\angle$. (?)

Therefore, AB is tangent to $\odot$ BC. (?)

In like manner, AC is tangent to $\odot$ BC.

PROP. XLIV. PROBLEM.

226. *Upon a given straight line, to describe a segment which shall contain a given angle.*



Given line AB, and $\angle$ A'.

Required to describe upon AB a segment such that every $\angle$ inscribed in the segment shall equal $\angle$ A'.

Construction. Construct $\angle$ $BAC = \angle$ A'. (§ 208)

Draw line $DE \perp$ to AB at its middle point. (§ 205)

Draw line $AF \perp AC$, intersecting DE at O.

With O as a centre and OA as a radius, describe $\odot$ $AMBN$.

Then, AMB will be the required segment.

Proof. Let AGB be any $\angle$ inscribed in segment AMB.

Then, $\angle$ AGB is measured by $\frac{1}{2}$ arc ANB. (?)

But, by cons., $AC \perp OA$.

Whence, AC is tangent to $\odot$ AMB. (?)

Therefore, $\angle$ BAC is measured by $\frac{1}{2}$ arc ANB. (§ 197)

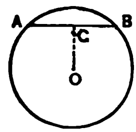
$$\therefore \angle AGB = \angle BAC = \angle A'. \quad (?)$$

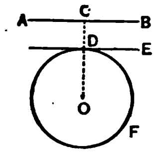
Hence, every $\angle$ inscribed in segment AMB equals $\angle$ A'. (§ 194)

EXERCISES.

85. Given the middle point of a chord of a circle, to construct the chord.

(To draw through C a chord which is bisected at C.)

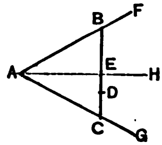




86. To draw a line tangent to a given circle and parallel to a given straight line.

(To draw a tangent ∥ AB.)

87. To draw a line tangent to a given circle and perpendicular to a given straight line.



88. To draw a straight line through a given point within a given acute ∠, forming with the sides of the angle an isosceles triangle.

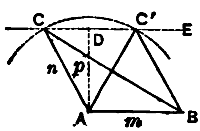
89. Given the base, an adjacent angle, and the altitude of a triangle, to construct the triangle.

(Draw a ∥ to the base at a distance equal to the altitude.)

90. Given the base, an adjacent side, and the altitude of a triangle, to construct the triangle.

Discuss the problem for the following cases:

1. $n > p$. 2. $n = p$. 3. $n < p$.



91. To construct a rhombus, having given its base and altitude.

(Draw a ∥ to the base at a distance equal to the altitude.)

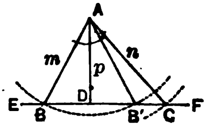
What restriction is there on the values of the given lines?

92. Given the altitude and the sides including the vertical angle of a triangle, to construct the triangle.

What restriction is there on the values of the given lines?

Discuss the problem for the following cases:

1. $m < n$ or $> n$. 2. $m = n$.



93. Given the altitude of a triangle, and the angles at the extremities of the base, to construct the triangle.

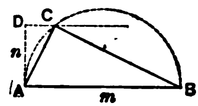
(The ∠ between the altitude and an adjacent side is the complement of the ∠ at the extremity of the base, if acute, or of its supplement, if obtuse.)

94. To construct an isosceles triangle, having given the base and the radius of the circumscribed circle.

What restriction is there on the values of the given lines?

95. To construct a square, having given one of its diagonals. (§ 195.)

96. To construct a right triangle, having given the hypotenuse and the length of the perpendicular drawn to it from the vertex of the right angle.

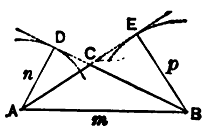


What restriction is there on the values of the given lines?

97. To construct a right triangle, having given the hypotenuse and a leg.

What restriction is there on the values of the given lines?

98. Given the base of a triangle and the perpendiculars from its extremities to the other sides, to construct the triangle. (§ 225.)



What restriction is there on the values of the given lines?

99. To describe a circle of given radius tangent to two given intersecting lines.

(Draw a ∥ to one of the lines at a distance equal to the radius.)

100. To describe a circle tangent to a given straight line, having its centre at a given point without the line.

101. To construct a circle having its centre in a given line, and passing through two given points without the line. (§ 163.)

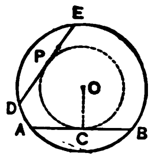
What restriction is there on the positions of the given points?

102. In a given straight line to find a point equally distant from two given intersecting lines. (§ 101.)

103. Given a side and the diagonals of a parallelogram, to construct the parallelogram.

What restriction is there on the values of the given lines?

104. Through a given point without a given straight line, to describe a circle tangent to the given line at a given point. (§ 163.)

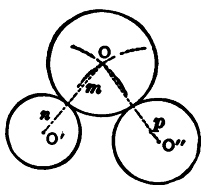


105. Through a given point within a circle to draw a chord equal to a given chord. (§ 164.)

What restriction is there on the position of the given point?

106. Through a given point to describe a circle of given radius tangent to a given straight line.

(Draw a $\parallel$ to the given line at a distance equal to the radius.)



107. To describe a circle of given radius tangent to two given circles.

(To describe a $\odot$ of radius m tangent to two given $\circledS$ whose radii are n and p, respectively.)

What restriction is there on the value of m?

108. To describe a circle tangent to two given parallels, and passing through a given point.

What restriction is there on the position of the given point?

109. To describe a circle of given radius, tangent to a given line and a given circle.

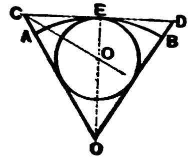
(Draw a $\parallel$ to the given line at a distance equal to the given radius.)

110. To construct a parallelogram, having given a side, an angle, and the diagonal drawn from the vertex of the angle.

111. In a given triangle to inscribe a rhombus, having one of its angles coincident with an angle of the triangle.

(Bisect the $\angle$ which is common to the $\triangle$ and the rhombus.)

112. To describe a circle touching two given intersecting lines, one of them at a given point. (§ 169.)



113. In a given sector to inscribe a circle.

(The problem is the same as inscribing a $\odot$ in $\triangle O'CD$.)

114. In a given right triangle to inscribe a square, having one of its angles coincident with the right angle of the triangle.

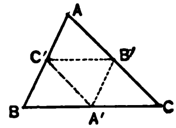
115. Through a vertex of a triangle to draw a straight line equally distant from the other vertices.

116. Given the base, the altitude, and the vertical angle of a triangle, to construct the triangle. (§ 226.)

(Construct on the given base as a chord a segment which shall contain the given ∠.)

117. Given the base of a triangle, its vertical angle, and the median drawn to the base, to construct the triangle.

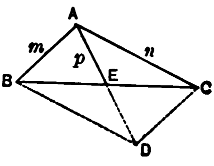
118. To construct a triangle, having given the middle points of its sides.



119. Given two sides of a triangle, and the median drawn to the third side, to construct the triangle.

(Construct △ ABD with its sides equal to m, n, and $2p$, respectively.)

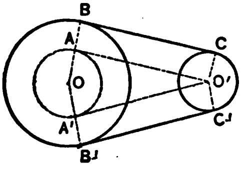
What restriction is there on the values of the given lines?

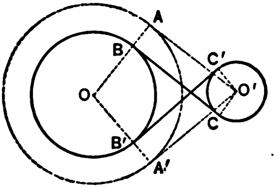


120. Given the base, the altitude, and the radius of the circumscribed circle of a triangle, to construct the triangle.

(The centre of the circumscribed ⊙ lies at a distance from each vertex equal to the radius of the ⊙.)

121. To draw common tangents to two given circles which do not intersect.





(To draw *exterior* common tangents, describe ⊙ AA' with its radius equal to the *difference* of the radii of the given Ⓢ.

To draw *interior* common tangents, describe ⊙ AA' with its radius equal to the *sum* of the radii of the given Ⓢ.)

Note. For additional exercises on Book II., see p. 224.

Book III.

THEORY OF PROPORTION.—SIMILAR POLYGONS.

DEFINITIONS.

227. A *Proportion* is a statement that two ratios are equal.

228. The statement that the ratio of a to b is equal to the ratio of c to d, may be written in either of the forms

$$a : b = c : d, \text{ or } \frac{a}{b} = \frac{c}{d}.$$

229. The first and fourth terms of a proportion are called the *extremes*, and the second and third terms the *means*.

The first and third terms are called the *antecedents*, and the second and fourth terms the *consequents*.

Thus, in the proportion $a : b = c : d$, a and d are the extremes, b and c the means, a and c the antecedents, and b and d the consequents.

230. If the means of a proportion are equal, either mean is called a *mean proportional* between the first and last terms, and the last term is called a *third proportional* to the first and second terms.

Thus, in the proportion $a : b = b : c$, b is a mean proportional between a and c, and c a third proportional to a and b.

231. A *fourth proportional* to three quantities is the fourth term of a proportion, whose first three terms are the three quantities taken in their order.

Thus, in the proportion $a : b = c : d$, d is a fourth proportional to a, b, and c.

PROP. I. THEOREM.

232. *In any proportion, the product of the extremes is equal to the product of the means.*

Given the proportion $a : b = c : d$.

To Prove $ad = bc$.

Proof. By § 228, $\frac{a}{b} = \frac{c}{d}$.

Multiplying both members of this equation by bd,

$$ad = bc.$$

233. Cor. *The mean proportional between two quantities is equal to the square root of their product.*

Given the proportion $a : b = b : c$. (1)

To Prove $b = \sqrt{ac}$.

Proof. From (1), $b^2 = ac$. (§ 232)

$$\therefore\ b = \sqrt{ac}.$$

PROP. II. THEOREM.

234. (Converse of Prop. I.) *If the product of two quantities is equal to the product of two others, one pair may be made the extremes, and the other pair the means, of a proportion.*

Given $ad = bc$. (1)

To Prove $a : b = c : d$.

Proof. Dividing both members of (1) by bd,

$$\frac{ad}{bd} = \frac{bc}{bd}.$$

Or, $\frac{a}{b} = \frac{c}{d}$.

Then by § 228, $a : b = c : d$.

In like manner, $a : c = b : d$,

$b : a = d : c$, etc.

PROP. III. THEOREM.

235. *In any proportion, the terms are in proportion by* ALTERNATION; *that is, the first term is to the third as the second term is to the fourth.*

Given the proportion $a : b = c : d.$ (1)

To Prove $a : c = b : d.$

Proof. From (1), $ad = bc.$ (§ 232)

$\therefore a : c = b : d.$ (§ 234)

PROP. IV. THEOREM.

236. *In any proportion, the terms are in proportion by* INVERSION; *that is, the second term is to the first as the fourth term is to the third.*

Given the proportion $a : b = c : d.$ (1)

To Prove $b : a = d : c.$

Proof. From (1), $ad = bc.$ (?)

$\therefore b : a = d : c.$ (?)

PROP. V. THEOREM.

237. *In any proportion, the terms are in proportion by* COMPOSITION; *that is, the sum of the first two terms is to the first term as the sum of the last two terms is to the third term.*

Given the proportion $a : b = c : d.$ (1)

To Prove $a + b : a = c + d : c.$

Proof. From (1), $ad = bc.$ (?)

Adding both members of the equation to ac,

$$ac + ad = ac + bc.$$

Factoring, $a(c + d) = c(a + b).$

$\therefore a + b : a = c + d : c.$ (§ 234)

In like manner, $a + b : b = c + d : d.$

Prop. VI. Theorem.

238. *In any proportion, the terms are in proportion by* Division; *that is, the difference of the first two terms is to the first term as the difference of the last two terms is to the third term.*

Given the proportion $a : b = c : d,$ (1)

in which $a > b$ and $c > d.$

To Prove $a - b : a = c - d : c.$

Proof. From (1), $ad = bc.$ (?)

Subtracting both members of the equation from ac,

$$ac - ad = ac - bc.$$

Factoring, $a(c - d) = c(a - b).$

$$\therefore a - b : a = c - d : c. \qquad (?)$$

In like manner, $a - b : b = c - d : d.$

Prop. VII. Theorem.

239. *In any proportion, the terms are in proportion by* Composition and Division; *that is, the sum of the first two terms is to their difference as the sum of the last two terms is to their difference.*

Given the proportion $a : b = c : d,$ (1)

in which $a > b$ and $c > d.$

To Prove $a + b : a - b = c + d : c - d.$

Proof. From (1), $\frac{a+b}{a} = \frac{c+d}{c},$ (§ 237)

and $\frac{a-b}{a} = \frac{c-d}{c}.$ (§ 238)

Dividing the first equation by the second,

$$\frac{a+b}{a-b} = \frac{c+d}{c-d}.$$

$$\therefore a + b : a - b = c + d : c - d.$$

Prop. VIII. Theorem.

240. *In a series of equal ratios, the sum of all the antecedents is to the sum of all the consequents as any antecedent is to its consequent.*

Given $a:b=c:d=e:f.$ (1)

To Prove $a+c+e:b+d+f=a:b.$

Proof. We have $ba=ab.$

Also, from (1), $bc=ad,$

and $be=af.$ (?)

Adding, $ba+bc+be=ab+ad+af.$

$\therefore\ b(a+c+e)=a(b+d+f).$

$\therefore\ a+c+e:b+d+f=a:b.$ (?)

Prop. IX. Theorem.

241. *In any proportion, like powers or like roots of the terms are in proportion.*

Given the proportion $a:b=c:d.$ (1)

To Prove $a^n:b^n=c^n:d^n.$

Proof. From (1), $\frac{a}{b}=\frac{c}{d}.$

Raising both members to the nth power,

$$\frac{a^n}{b^n}=\frac{c^n}{d^n}.$$

$$\therefore\ a^n:b^n=c^n:d^n.$$

In like manner, $\sqrt[n]{a}:\sqrt[n]{b}=\sqrt[n]{c}:\sqrt[n]{d}.$

Note. The ratio of two magnitudes of the same kind is equal to the ratio of their numerical measures when referred to a common unit (§ 183); hence, in any proportion involving the ratio of two magnitudes of the same kind, we may regard the ratio of the magnitudes as replaced by the ratio of their numerical measures when referred to a common unit.

Thus, let AB, CD, EF, and GH be four lines such that

$$AB:CD=EF:GH.$$

Then, $AB\times GH=CD\times EF.$ (§ 232)

This means that the product of the *numerical measures* of AB and GH is equal to the product of the *numerical measures* of CD and EF.

An interpretation of this nature must be given to all applications of §§ 232, 233 and 241.

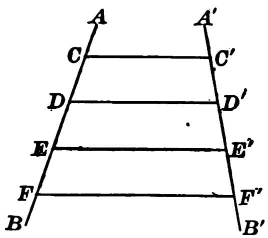
EXERCISES.

1. Find a fourth proportional to 65, 80, and 91.
2. Find a mean proportional between 28 and 63.
3. Find a third proportional to $\frac{3}{4}$ and $\frac{5}{6}$.
4. What is the second term of a proportion whose first, third, and fourth terms are 45, 160, and 224, respectively?

PROPORTIONAL LINES.

Prop. X. Theorem.

242. *If a series of parallels, cutting two straight lines, intercept equal distances on one of these lines, they also intercept equal distances on the other.*



Given lines AB and $A'B'$ cut by ∥s CC', DD', EE', and FF' at points C, D, E, F, and C', D', E', F', respectively, so that $CD = DE = EF$.

To Prove $C'D' = D'E' = E'F'$.

Proof. In trapezoid $CC'E'E$, by hyp., DD' is ∥ to the bases, and bisects CE; it therefore bisects $C'E'$. (§ 133)

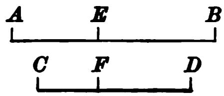
$$\therefore C'D' = D'E'. \qquad (1)$$

In like manner, in trapezoid $DD'F'F$, EE' is ∥ to the bases, and bisects DF.

$$\therefore D'E' = E'F'. \qquad (2)$$

From (1) and (2), $C'D' = D'E' = E'F'$. (?)

243. Def. Two straight lines are said to be *divided proportionally* when their corresponding segments are in the same ratio as the lines themselves.



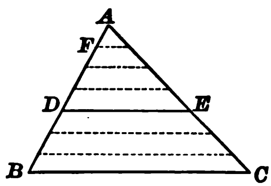
Thus, lines AB and CD are divided proportionally if

$$\frac{AE}{CF}=\frac{BE}{DF}=\frac{AB}{CD}.$$

Prop. XI. Theorem.

244. *A parallel to one side of a triangle divides the other two sides proportionally.*

Case I. *When the segments of each side are commensurable.*



Given, in $\triangle ABC$, segments AD and BD of side AB commensurable, and line $DE \parallel BC$, meeting AC at E.

To Prove $$\frac{AD}{BD}=\frac{AE}{CE}.$$

Proof. Let AF be a common measure of AD and BD; and let it be contained 4 times in AD, and 3 times in BD.

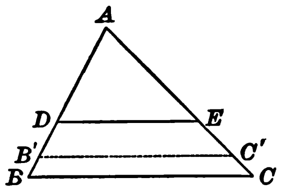
$$\therefore \frac{AD}{BD}=\frac{4}{3}. \qquad (1)$$

Drawing ∥s to BC through the several points of division of AB, AE will be divided into 4 parts, and CE into 3 parts, all of which parts are equal. (§ 242)

$$\therefore \frac{AE}{CE}=\frac{4}{8}. \qquad (2)$$

From (1) and (2), $$\frac{AD}{BD}=\frac{AE}{CE}. \qquad (?)$$

Case II. *When the segments of each side are incommensurable.*



Given, in $\triangle ABC$, segments AD and BD of side AB incommensurable, and line $DE \parallel BC$, meeting AC at E.

To Prove $\dfrac{AD}{BD} = \dfrac{AE}{CE}$.

Proof. Let AD be divided into any number of equal parts, and let one of these parts be applied to BD as a unit of measure.

Since AD and BD are incommensurable, a certain number of the equal parts will extend from D to B', leaving a remainder $BB' <$ one of the equal parts.

Draw line $B'C' \parallel BC$, meeting AC at C'.

Then, since AD and $B'D$ are commensurable,

$$\frac{AD}{B'D} = \frac{AE}{C'E}. \qquad \text{(§ 244, Case I.)}$$

Now let the number of subdivisions of AD be indefinitely increased.

Then the unit of measure will be indefinitely diminished, and the remainder BB' will approach the limit 0.

Then, $\dfrac{AD}{B'D}$ will approach the limit $\dfrac{AD}{BD}$,

and $\dfrac{AE}{C'E}$ will approach the limit $\dfrac{AE}{CE}$.

By the Theorem of Limits, these limits are equal. (?)

$$\therefore \frac{AD}{BD} = \frac{AE}{CE}.$$

245. Cor. I. We may write the result of § 244,

$$AD : BD = AE : CE. \qquad (1)$$

$$\therefore AD + BD : AD = AE + CE : AE. \qquad (§ 237)$$

$$\therefore AB : AD = AC : AE. \qquad (2)$$

In like manner, $AB : BD = AC : CE.$ (3)

246. Cor. II. From (2), § 245,

$$AB : AC = AD : AE,$$

and from (3), $AB : AC = BD : CE.$ (§ 235)

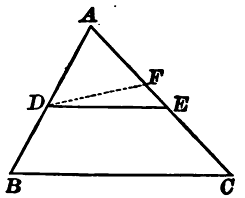
Then, by Ax. 1, $\dfrac{AB}{AC} = \dfrac{AD}{AE} = \dfrac{BD}{CE}.$ (4)

247. Sch. The proportions (1), (2), (3), and (4), of §§ 245 and 246, are all included in the general statement,

A parallel to one side of a triangle divides the other two sides proportionally.

PROP. XII. THEOREM.

248. (Converse of Prop. XI.) *A line which divides two sides of a triangle proportionally is parallel to the third side.*



Given, in $\triangle ABC$, line DE meeting AB and AC at D and E respectively, so that

$$\frac{AB}{AD} = \frac{AC}{AE}.$$

To Prove $DE \parallel BC.$

Proof. If DE is not $\parallel BC$, draw line $DF \parallel BC$, meeting AC at F.

$$\therefore \frac{AB}{AD} = \frac{AC}{AF}. \qquad (§ 247)$$

But by hyp., $$\frac{AB}{AD}=\frac{AC}{AE}.$$

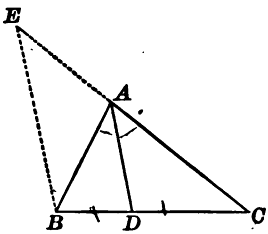
$$\therefore \frac{AC}{AE}=\frac{AC}{AF}. \qquad (?)$$

$$\therefore AE = AF.$$

Then, DF coincides with DE, and $DE \parallel BC$. (Ax. 3)

Prop. XIII. Theorem.

249. *In any triangle, the bisector of an angle divides the opposite side into segments proportional to the adjacent sides.*



Given line AD bisecting $\angle A$ of $\triangle ABC$, meeting BC at D.

To Prove $$\frac{DB}{DC}=\frac{AB}{AC}.$$

Proof. Draw line $BE \parallel AD$, meeting CA produced at E.

Then, since $\parallel$s AD and BE are cut by AB,

$$\angle ABE = \angle BAD. \qquad (?)$$

And since $\parallel$s AD and BE are cut by CE,

$$\angle AEB = \angle CAD. \qquad (?)$$

But by hyp., $$\angle BAD = \angle CAD.$$

$$\therefore \angle ABE = \angle AEB. \qquad (?)$$

$$\therefore AB = AE. \qquad (?)$$

Now since AD is $\parallel$ to side BE of $\triangle BCE$,

$$\frac{DB}{DC}=\frac{AE}{AC}. \qquad (\S\ 247)$$

Putting for AE its equal AB, we have

$$\frac{DB}{DC}=\frac{AB}{AC}.$$

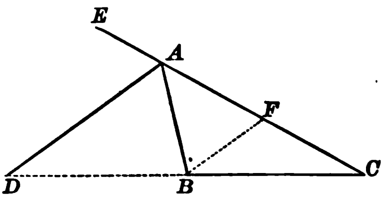
250. Def. The *segments* of a line by a point are the distances from the point to the extremities of the line, whether the point is in the line itself, or in the line produced.

omit

PROP. XIV. THEOREM.

251. *In any triangle the bisector of an exterior angle divides the opposite side externally into segments proportional to the adjacent sides.*

Note. The theorem does not hold for the exterior angle at the vertex of an *isosceles* triangle.



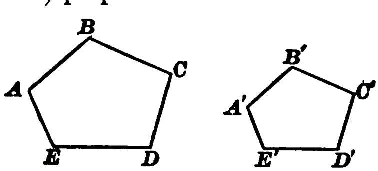
Given line AD bisecting ext. $\angle BAE$ of $\triangle ABC$, meeting CB produced at D.

To Prove $$\frac{DB}{DC}=\frac{AB}{AC}.$$

(Draw $BF \parallel AD$; then $\angle ABF = \angle AFB$, and $AF = AB$; BF is $\parallel$ to side AD of $\triangle ACD$.)

SIMILAR POLYGONS.

252. Def. Two polygons are said to be *similar* if they are mutually equiangular (§ 122), and have their homologous sides (§ 123) proportional.



Thus, polygons $ABCDE$ and $A'B'C'D'E'$ are similar if

$$\angle A = \angle A',\ \angle B = \angle B', \text{ etc.},$$

and, $$\frac{AB}{A'B'}=\frac{BC}{B'C'}=\frac{CD}{C'D'}, \text{ etc.}$$

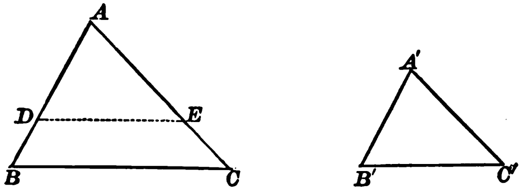
253. Sch. The following are given for reference:

1. *In similar polygons, the homologous angles are equal.*

2. *In similar polygons, the homologous sides are proportional.*

PROP. XV. THEOREM.

254. *Two triangles are similar when they are mutually equiangular.*



Given, in ⊿s ABC and $A'B'C'$,

$$\angle A = \angle A', \ \angle B = \angle B', \text{ and } \angle C = \angle C'.$$

To Prove ⊿s ABC and $A'B'C'$ similar.

Proof. Place $\triangle A'B'C'$ in the position ADE; $\angle A'$ coinciding with its equal $\angle A$, vertices B' and C' falling at D and E, respectively, and side $B'C'$ at DE.

Since, by hyp., $\angle ADE = \angle B$, $DE \parallel BC$. (?)

$$\therefore \frac{AB}{AD} = \frac{AC}{AE}. \qquad (\S\ 247)$$

That is,
$$\frac{AB}{A'B'} = \frac{AC}{A'C'}. \qquad (1)$$

In like manner, by placing $\triangle A'B'C'$ so that $\angle B'$ shall coincide with its equal $\angle B$, vertices A' and C' falling on sides AB and BC, respectively, we may prove

$$\frac{AB}{A'B'} = \frac{BC}{B'C'}. \qquad (2)$$

From (1) and (2),
$$\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}. \qquad (?)$$

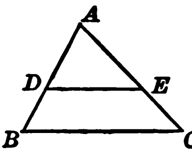
Then, ⊿s ABC and $A'B'C'$ have their homologous sides proportional, and are similar. (§ 252)

255. Cor. I. *Two triangles are similar when two angles of one are equal respectively to two angles of the other.*

For their remaining ∡ are equal each to each. (§ 86)

256. Cor. II. *Two right triangles are similar when an acute angle of one is equal to an acute angle of the other.*

257. Cor. III. *If a line be drawn between two sides of a triangle parallel to the third side, the triangle formed is similar to the given triangle.*



Given line $DE \parallel$ to side BC of $\triangle ABC$, meeting AB and AC at D and E, respectively.

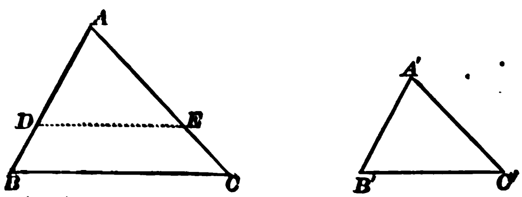
To Prove $\triangle ADE$ similar to $\triangle ABC$.

(The ∆ are mutually equiangular.)

258. Sch. *In similar triangles, the homologous sides lie opposite the equal angles.*

Prop. XVI. Theorem.

259. *Two triangles are similar when their homologous sides are proportional.*



Given, in ∆ ABC and $A'B'C'$,

$$\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}.$$

To Prove ∆ ABC and $A'B'C'$ similar.

Proof. On AB and AC, take $AD = A'B'$ and $AE = A'C'$. Draw line DE; then, from the given proportion,

$$\frac{AB}{AD} = \frac{AC}{AE}.$$

$$\therefore DE \parallel BC. \qquad (\S\ 248)$$

Then, $\triangle$ ADE and ABC are similar. (§ 257)

$$\therefore \frac{AB}{AD} = \frac{BC}{DE}, \text{ or } \frac{AB}{A'B'} = \frac{BC}{DE}. \qquad (\S\ 253, 2)$$

But by hyp., $$\frac{AB}{A'B'} = \frac{BC}{B'C'}.$$

$$\therefore DE = B'C'.$$

$$\therefore \triangle ADE = \triangle A'B'C'. \qquad (\S\ 69)$$

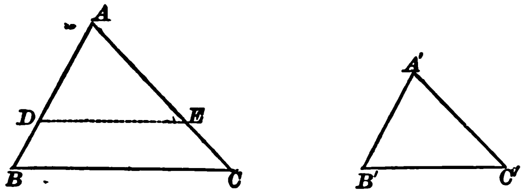
But, $\triangle ADE$ has been proved similar to $\triangle ABC$.

Hence, $\triangle A'B'C'$ is similar to $\triangle ABC$.

260. Sch. To prove that two polygons in general are similar, it must be shown that they are mutually equiangular, and have their homologous sides proportional (§ 252); but in the case of two *triangles*, each of these conditions involves the other (§§ 254, 259), so that it is only necessary to show that one of the tests of similarity is satisfied.

PROP. XVII. THEOREM.

261. *Two triangles are similar when they have an angle of one equal to an angle of the other, and the sides including these angles proportional.*



Given, in $\triangle$ ABC and $A'B'C'$,

$$\angle A = \angle A', \text{ and } \frac{AB}{A'B'} = \frac{AC}{A'C'}.$$

To Prove $\triangle$ ABC and $A'B'C'$ similar.

(Place $\triangle A'B'C'$ in the position ADE; by § 248, $DE \parallel BC$; the theorem follows by § 257.)

Prop. XVIII. Theorem.

262. *Two triangles are similar when their sides are parallel each to each, or perpendicular each to each.*

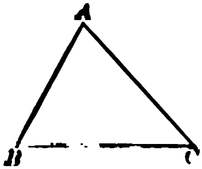


Fig. 1.

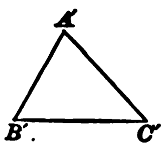


Fig. 2.

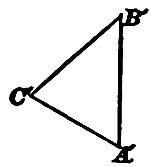


Fig. 3.

Given sides AB, AC, and BC, of $\triangle ABC$, $\parallel$ respectively to sides $A'B'$, $A'C'$, and $B'C'$ of $\triangle A'B'C'$ in Fig. 2, and $\perp$ respectively to sides $A'B'$, $A'C'$, and $B'C'$ of $\triangle A'B'C'$ in Fig. 3.

To Prove $\triangle$s ABC and $A'B'C'$ similar.

Proof. Since the sides of $\angle$s A and A' are $\parallel$ each to each, or $\perp$ each to each, $\angle$s A and A' are either equal or supplementary. (§§ 81, 82, 83)

In like manner, $\angle$s B and B', and $\angle$s C and C', are either equal or supplementary.

We may then make the following hypotheses with regard to the $\angle$s of the $\triangle$s:

1. $A + A' = 2$ rt. $\angle$s, $B + B' = 2$ rt. $\angle$s, $C + C' = 2$ rt. $\angle$s.
2. $A + A' = 2$ rt. $\angle$s, $B + B' = 2$ rt. $\angle$s, $C = C'$.
3. $A + A' = 2$ rt. $\angle$s, $B = B'$, $C + C' = 2$ rt. $\angle$s.
4. $A = A'$, $B + B' = 2$ rt. $\angle$s, $C + C' = 2$ rt. $\angle$s.
5. $A = A'$, $B = B'$, whence $C = C'$. (§ 86)

The first four hypotheses are impossible; for, in either case, the sum of the six $\angle$s of the two $\triangle$s would be >4 rt. $\angle$s. (§ 84)

We can then have only $A = A'$, $B = B'$, and $C = C'$.

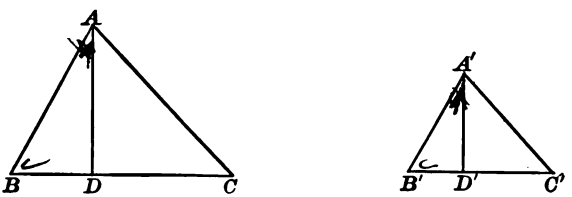
Therefore, $\triangle$s ABC and $A'B'C'$ are similar. (§ 254)

263. Sch. 1. *In similar triangles whose sides are parallel each to each, the parallel sides are homologous.*

2. *In similar triangles whose sides are perpendicular each to each, the perpendicular sides are homologous.*

Prop. XIX. Theorem.

264. *The homologous altitudes of two similar triangles are in the same ratio as any two homologous sides.*



Given AD and $A'D'$ homologous altitudes of similar $\triangle$ ABC and $A'B'C'$.

To Prove $\dfrac{AD}{A'D'} = \dfrac{AB}{A'B'} = \dfrac{AC}{A'C'} = \dfrac{BC}{B'C'}$.

(Rt. $\triangle$ ABD and $A'B'D'$ are similar by § 256.)

265. Sch. In two similar triangles, any two homologous lines are in the same ratio as any two homologous sides.

EXERCISES.

5. The sides of a triangle are $AB = 8$, $BC = 6$, and $CA = 7$; find the segments into which each side is divided by the bisector of the opposite angle.

6. The sides of a triangle are $AB = 5$, $BC = 7$, and $CA = 8$; find the segments into which each side is divided by the bisector of the exterior angle at the opposite vertex.

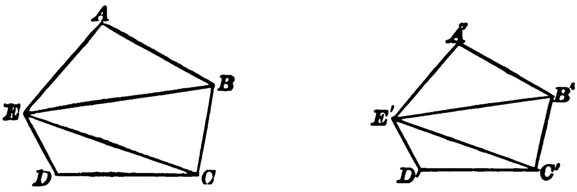
7. The sides of a triangle are 5, 7, and 9. The shortest side of a similar triangle is 14. What are the other two sides?

8. Two isosceles triangles are similar when their vertical angles are equal. (§ 255.)

9. The base and altitude of a triangle are 5 ft. 10 in. and 3 ft. 6 in., respectively. If the homologous base of a similar triangle is 7 ft. 6 in., find its homologous altitude.

Prop. XX. Theorem.

266. *Two polygons are similar when they are composed of the same number of triangles, similar each to each, and similarly placed.*



Given, in polygons AC and $A'C'$, $\triangle ABE$ similar to $\triangle A'B'E'$, $\triangle BCE$ to $\triangle B'C'E'$, and $\triangle CDE$ to $\triangle C'D'E'$.

To Prove polygons AC and $A'C'$ similar.

Proof. Since $\triangle$s ABE and $A'B'E'$ are similar,

$$\angle A = \angle A'. \qquad (?)$$

Also,
$$\angle ABE = \angle A'B'E'.$$

And since $\triangle$s BCE and $B'C'E'$ are similar,

$$\angle EBC = \angle E'B'C'.$$

$$\therefore \angle ABE + \angle EBC = \angle A'B'E' + \angle E'B'C'.$$

Or,
$$\angle ABC = \angle A'B'C'.$$

In like manner, $\angle BCD = \angle B'C'D'$, etc.

Then, AC and $A'C'$ are mutually equiangular.

Again, since $\triangle ABE$ is similar to $\triangle A'B'E'$, and $\triangle BCE$ to $\triangle B'C'E'$,

$$\frac{AB}{A'B'} = \frac{BE}{B'E'} \text{ and } \frac{BE}{B'E'} = \frac{BC}{B'C'}. \qquad (?)$$

$$\therefore \frac{AB}{A'B'} = \frac{BC}{B'C'}. \qquad (?)$$

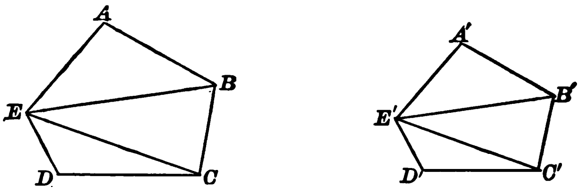
In like manner,
$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'}, \text{ etc.}$$

Then, AC and $A'C'$ have their homologous sides proportional.

Therefore, AC and $A'C'$ are similar. **(§ 252)**

Prop. XXI. Theorem.

267. (Converse of Prop. XX.) *Two similar polygons may be decomposed into the same number of triangles, similar each to each, and similarly placed.*



Given E and E' homologous vertices of similar polygons AC and $A'C'$, and lines EB, EC, $E'B'$, and $E'C'$.

To Prove $\triangle\, ABE$ similar to $\triangle\, A'B'E'$, $\triangle\, BCE$ to $\triangle\, B'C'E'$, and $\triangle\, CDE$ to $\triangle\, C'D'E'$.

Proof. Since polygons AC and $A'C'$ are similar,

$$\angle A = \angle A' \text{ and } \frac{AE}{A'E'} = \frac{AB}{A'B'}. \qquad (?)$$

Then, $\triangle$s ABE and $A'B'E'$ are similar. (§ 261)

Again, since the polygons are similar,

$$\angle ABC = \angle A'B'C'.$$

And since $\triangle$s ABE and $A'B'E'$ are similar,

$$\angle ABE = \angle A'B'E'.$$

$$\therefore\ \angle ABC - \angle ABE = \angle A'B'C' - \angle A'B'E'.$$

Or, $$\angle EBC = \angle E'B'C'.$$

Also, since the polygons are similar, $\dfrac{AB}{A'B'} = \dfrac{BC}{B'C'}$.

And since $\triangle$s ABE and $A'B'E'$ are similar, $\dfrac{AB}{A'B'} = \dfrac{BE}{B'E'}$.

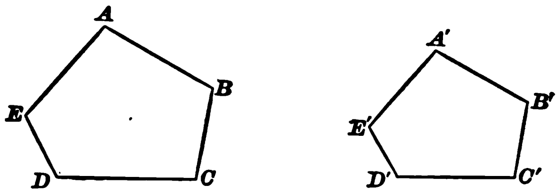
$$\therefore\ \frac{BC}{B'C'} = \frac{BE}{B'E'}. \qquad (?)$$

Then, since $\angle EBC = \angle E'B'C'$, and $\dfrac{BC}{B'C'} = \dfrac{BE}{B'E'}$, $\triangle$s BCE and $B'C'E'$ are similar. (?)

In like manner, we may prove $\triangle$s CDE and $C'D'E'$ similar.

Prop. XXII. Theorem.

268. *The perimeters of two similar polygons are in the same ratio as any two homologous sides.*



Given AB and $A'B'$, BC and $B'C'$, CD and $C'D'$, etc., homologous sides of similar polygons AC and $A'C'$.

To Prove

$$\frac{AB + BC + CD + \text{etc.}}{A'B' + B'C' + C'D' + \text{etc.}} = \frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'}, \text{ etc.}$$

(Apply § 240 to the equal ratios of § 252.)

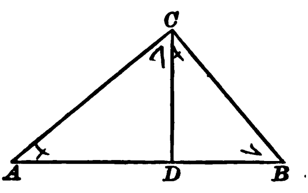
Prop. XXIII. Theorem.

269. *If a perpendicular be drawn from the vertex of the right angle to the hypotenuse of a right triangle,*

I. *The triangles formed are similar to the whole triangle, and to each other.*

II. *The perpendicular is a mean proportional between the segments of the hypotenuse.*

III. *Either leg is a mean proportional between the whole hypotenuse and the adjacent segment.*



Given line $CD \perp$ hypotenuse AB of rt. $\triangle ABC$.

I. **To Prove** $\triangle$s ACD and BCD similar to $\triangle ABC$, and to each other.

Proof. In rt. $\triangle$s ACD and ABC,

$$\angle A = \angle A.$$

Then, $\triangle ACD$ is similar to $\triangle ABC$. (§ 256)

In like manner, $\triangle BCD$ is similar to $\triangle ABC$.

Then, $\triangle$s ACD and BCD are similar to each other, for each is similar to $\triangle ABC$.

II. **To Prove** $AD : CD = CD : BD$.

Proof. Since $\triangle$s ACD and BCD are similar,

$$\angle ACD = \angle B \text{ and } \angle A = \angle BCD. \quad (§ 253, 1)$$

In $\triangle$s ACD and BCD, AD and CD are homologous sides, for they lie opposite the equal $\angle$s ACD and B, respectively; also, CD and BD are homologous sides, for they lie opposite the equal $\angle$s A and BCD, respectively. (§ 258)

$$\therefore AD : CD = CD : BD. \quad (?)$$

III. **To Prove** $AB : AC = AC : AD$.

Proof. Since $\triangle$s ABC and ACD are similar,

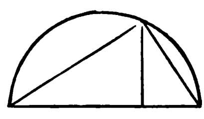
$$\angle ACB = \angle ADC \text{ and } \angle B = \angle ACD. \quad (?)$$

In $\triangle$s ABC and ACD, AB and AC are homologous sides, for they lie opposite the equal $\angle$s ACB and ADC, respectively; also, AC and AD are homologous sides, for they lie opposite the equal $\angle$s B and ACD, respectively. (?)

$$\therefore AB : AC = AC : AD. \quad (?)$$

In like manner, $AB : BC = BC : BD$.

270. Cor. I. Since an angle inscribed in a semicircle is a right angle (§ 195), it follows that:



If a perpendicular be drawn from any point in the circumference of a circle to a diameter,

1. *The perpendicular is a mean proportional between the segments of the diameter.*

2. *The chord joining the point to either extremity of the diameter is a mean proportional between the whole diameter and the adjacent segment.*

271. Cor. II. The three proportions of § 269 give

$$\overline{CD}^2 = AD \times BD,$$
$$\overline{AC}^2 = AB \times AD,$$
and $$\overline{BC}^2 = AB \times BD. \qquad (?)$$

Hence, *if a perpendicular be drawn from the vertex of the right angle to the hypotenuse of a right triangle,*

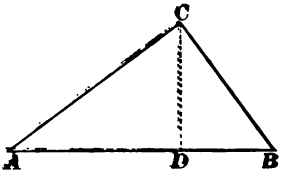
1. *The square of the perpendicular is equal to the product of the segments of the hypotenuse.*

2. *The square of either leg is equal to the product of the whole hypotenuse and the adjacent segment.*

As stated in Note, p. 126, these equations mean that the square of the *numerical measure* of CD is equal to the product of the *numerical measures* of AD and BD, etc.

Prop. XXIV. Theorem.

272. *In any right triangle, the square of the hypotenuse is equal to the sum of the squares of the legs.*



Given AB the hypotenuse of rt. $\triangle ABC$.

To Prove $\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2$.

Proof. Draw line $CD \perp AB$.

Then, $\overline{AC}^2 = AB \times AD$,

and $\overline{BC}^2 = AB \times BD$. (§ 271, 2)

Adding, $\overline{AC}^2 + \overline{BC}^2 = AB \times (AD + BD) = AB \times AB$.

$$\therefore \overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2.$$

273. Cor. I. It follows from § 272 that

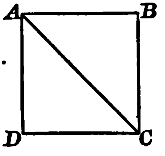
$$\overline{AC}^2 = \overline{AB}^2 - \overline{BC}^2, \text{ and } \overline{BC}^2 = \overline{AB}^2 - \overline{AC}^2.$$

That is, *in any right triangle, the square of either leg is equal to the square of the hypotenuse, minus the square of the other leg.*

274. Cor. II. If AC is a diagonal of square $ABCD$,

$$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2 = \overline{AB}^2 + \overline{AB}^2. \quad (\S\ 272)$$

$$\therefore\ \overline{AC}^2 = 2\,\overline{AB}^2.$$



Dividing both members by $\overline{AB}^2$,

$$\frac{\overline{AC}^2}{\overline{AB}^2} = 2, \text{ or } \frac{AC}{AB} = \sqrt{2}.$$

Hence, *the diagonal of a square is incommensurable with its side* (§ 181).

EXERCISES.

10. The perimeters of two similar polygons are 119 and 68; if a side of the first is 21, what is the homologous side of the second?

11. What is the length of the tangent to a circle whose diameter is 16, from a point whose distance from the centre is 17?

12. What is the length of the longest straight line which can be drawn on a floor 33 ft. 4 in. long, and 16 ft. 3 in. wide?

13. A ladder 32 ft. 6 in. long is placed so that it just reaches a window 26 ft. above the street; and when turned about its foot, just reaches a window 16 ft. 6 in. above the street on the other side. Find the width of the street.

14. The altitude of an equilateral triangle is 5; what is its side?

15. Find the length of the diagonal of a square whose side is 1 ft. 3 in.

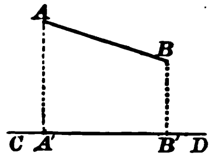
16. One of the non-parallel sides of a trapezoid is perpendicular to the bases. If the length of this side is 40, and of the parallel sides 31 and 22, respectively, what is the length of the other side?

17. The length of the tangent to a circle, whose diameter is 80, from a point without the circumference, is 42. What is the distance of the point from the centre?

18. If the length of the common chord of two intersecting circles is 16, and their radii are 10 and 17, what is the distance between their centres? (§ 178.)

DEFINITIONS.

275. The *projection of a point* upon a straight line of indefinite length, is the foot of the perpendicular drawn from the point to the line.



Thus, if line AA' be perpendicular to line CD, the projection of point A on line CD is point A'.

276. The *projection of a finite straight line* upon a straight line of indefinite length, is that portion of the second line included between the projections of the extremities of the first.

Thus, if lines AA' and BB' be perpendicular to line CD, the projection of line AB upon line CD is line $A'B'$.

PROP. XXV. THEOREM

277. *In any triangle, the square of the side opposite an acute angle is equal to the sum of the squares of the other two sides, minus twice the product of one of these sides and the projection of the other side upon it.*

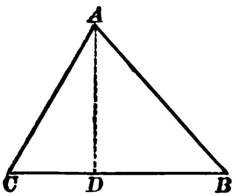


Fig. 1.

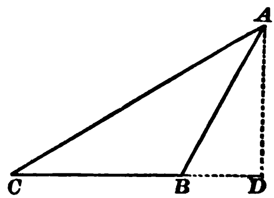


Fig. 2.

Given C an acute $\angle$ of $\triangle ABC$, and CD the projection of side AC upon side CB, produced if necessary. (§ 276)

To Prove $\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 - 2\,BC \times CD$.

Proof. Draw line AD; then, $AD \perp CD$. (§ 276)

There will be two cases according as D falls on CB (Fig. 1), or on CB produced (Fig. 2).

In Fig. 1, $BD = BC - CD.$

In Fig. 2, $BD = CD - BC.$

Squaring both members of the equation, we have by the algebraic rule for the square of the difference of two numbers, in either case,

$$\overline{BD}^2 = \overline{BC}^2 + \overline{CD}^2 - 2\,BC \times CD.$$

Adding $\overline{AD}^2$ to both members,

$$\overline{AD}^2 + \overline{BD}^2 = \overline{BC}^2 + \overline{AD}^2 + \overline{CD}^2 - 2\,BC \times CD.$$

But in rt. ▲ ABD and ACD,

$$\overline{AD}^2 + \overline{BD}^2 = \overline{AB}^2,$$

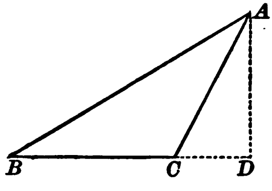
and $$\overline{AD}^2 + \overline{CD}^2 = \overline{AC}^2. \qquad (\S\ 272)$$

Substituting these values, we have

$$\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 - 2\,BC \times CD.$$

PROP. XXVI. THEOREM.

278. *In any triangle having an obtuse angle, the square of the side opposite the obtuse angle is equal to the sum of the squares of the other two sides, plus twice the product of one of these sides and the projection of the other side upon it.*



Given C an obtuse $\angle$ of $\triangle ABC$, and CD the projection of side AC upon side BC produced.

To Prove $\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 + 2\,BC \times CD.$

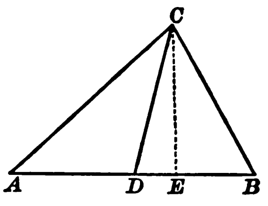
(We have $BD = BC + CD$; square both members, using the algebraic rule for the square of the sum of two numbers, and then add $\overline{AD}^2$ to both members.)

PROP. XXVII. THEOREM.

279. *In any triangle, if a median be drawn from the vertex to the base,*

I. *The sum of the squares of the other two sides is equal to twice the square of half the base, plus twice the square of the median.*

II. *The difference of the squares of the other two sides is equal to twice the product of the base and the projection of the median upon the base.*



Given DE the projection of median CD upon base AB of $\triangle ABC$; and $AC > BC$.

To Prove I. $\overline{AC}^2 + \overline{BC}^2 = 2\,\overline{AD}^2 + 2\,\overline{CD}^2$.

II. $\overline{AC}^2 - \overline{BC}^2 = 2\,AB \times DE$.

Proof. Since $AC > BC$, E falls between B and D.

Then, $\angle ADC$ is obtuse, and $\angle BDC$ acute.

Hence, in $\triangle$ ADC and BDC,

$$\overline{AC}^2 = \overline{AD}^2 + \overline{CD}^2 + 2\,AD \times DE, \qquad (\S\ 278)$$

and
$$\overline{BC}^2 = \overline{BD}^2 + \overline{CD}^2 - 2\,BD \times DE. \qquad (\S\ 277)$$

But by hyp., $BD = AD$.

$$\therefore\ \overline{AC}^2 = \overline{AD}^2 + \overline{CD}^2 + AB \times DE, \qquad (1)$$

and
$$\overline{BC}^2 = \overline{AD}^2 + \overline{CD}^2 - AB \times DE. \qquad (2)$$

Adding (1) and (2), we have

$$\overline{AC}^2 + \overline{BC}^2 = 2\,\overline{AD}^2 + 2\,\overline{CD}^2.$$

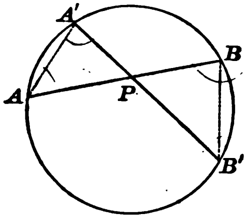
Subtracting (2) from (1), we have

$$\overline{AC}^2 - \overline{BC}^2 = 2\,AB \times DE.$$

X

PROP. XXVIII. THEOREM.

280. *If any two chords be drawn through a fixed point within a circle, the product of the segments of one chord is equal to the product of the segments of the other.*



Given AB and $A'B'$ any two chords passing through fixed point P within $\odot AA'B$.

To Prove $AP \times BP = A'P \times B'P$.

Proof. Draw lines AA' and BB'.

Then, in $\triangle AA'P$ and $BB'P$,

$$\angle A = \angle B',$$

for each is measured by $\frac{1}{2}$ arc $A'B$. (?)

In like manner, $\angle A' = \angle B$.

Then, $\triangle AA'P$ and $BB'P$ are similar. (?)

In similar $\triangle AA'P$ and $BB'P$, sides AP and $B'P$ are homologous, as also are sides $A'P$ and BP. (§ 258)

$$\therefore AP : A'P = B'P : BP. \quad (?)$$

$$\therefore AP \times BP = A'P \times B'P. \quad (?)$$

281. Sch. The proportion of § 280 may be written

$$\frac{AP}{A'P} = \frac{B'P}{BP}, \text{ or } \frac{AP}{A'P} = \frac{1}{\frac{BP}{B'P}}.$$

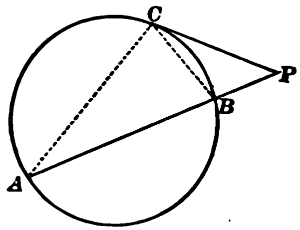
If two magnitudes, such as the segments of a chord passing through a fixed point, are so related that the ratio of any two values of one is equal to the *reciprocal* of the ratio of the corresponding values of the other, they are said to be *reciprocally proportional.*

Then, the theorem may be written,

If any two chords be drawn through a fixed point within a circle, their segments are reciprocally proportional.

PROP. XXIX. THEOREM.

282. *If through a fixed point without a circle a secant and a tangent be drawn, the product of the whole secant and its external segment is equal to the square of the tangent.*



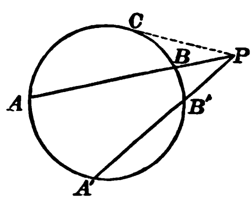
Given AP a secant, and CP a tangent, passing through fixed point P without $\odot ABC$.

To Prove $AP \times BP = \overline{CP}^2$.

($\angle A = \angle BCP$, for each is measured by $\frac{1}{2}$ arc BC (?); then $\triangle$ ACP and BCP are similar, and their homologous sides are proportional.)

283. Cor. I. *If through a fixed point without a circle a secant and a tangent be drawn, the tangent is a mean proportional between the whole secant and its external segment.*

284. Cor. II. *If any two secants be drawn through a fixed point without a circle, the product of one and its external segment is equal to the product of the other and its external segment.*



Given P any point without $\odot ABC$, and AP and $A'P$ secants intersecting the circumference at A and B, and A' and B', respectively.

To Prove $AP \times BP = A'P \times B'P$.

(We have $AP \times BP$ and $A'P \times B'P$ each equal to $\overline{CP}^2$.)

285. Cor. III. *If any two secants be drawn through a fixed point without a circle, the whole secants and their external segments are reciprocally proportional* (§ 281).

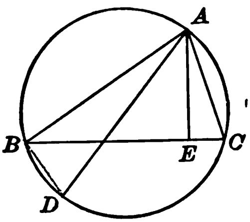
EXERCISES.

19. Find the length of the common tangent to two circles whose radii are 11 and 18, if the distance between their centres is 25.

20. AB is the hypotenuse of right triangle ABC. If perpendiculars be drawn to AB at A and B, meeting AC produced at D, and BC produced at E, prove triangles ACE and BCD similar.

Prop. XXX. Theorem.

286. *In any triangle, the product of any two sides is equal to the diameter of the circumscribed circle, multiplied by the perpendicular drawn to the third side from the vertex of the opposite angle.*



Given AD a diameter of the circumscribed $\odot ACD$ of $\triangle ABC$, and line $AE \perp BC$.

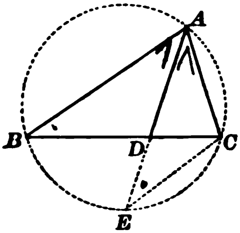
To Prove $AB \times AC = AD \times AE$.

(In rt. $\triangle$s ABD and ACE, $\angle D = \angle C$; then, the $\triangle$s are similar, and their homologous sides are proportional.)

287. Cor. *In any triangle, the diameter of the circumscribed circle is equal to the product of any two sides divided by the perpendicular drawn to the third side from the vertex of the opposite angle.*

PROP. XXXI. THEOREM.

288. *In any triangle, the product of any two sides is equal to the product of the segments of the third side formed by the bisector of the opposite angle, plus the square of the bisector.*



Given, in $\triangle ABC$, line AD bisecting $\angle A$, meeting BC at D.

To Prove $AB \times AC = BD \times DC + \overline{AD}^2$.

Proof. Circumscribe a $\odot$ about $\triangle ABC$; produce AD to meet the circumference at E, and draw line CE.

Then in $\triangle$ ABD and ACE, by hyp.,

$$\angle BAD = \angle CAE.$$

Also, $\angle B = \angle E$,

since each is measured by $\frac{1}{2}$ arc AC. (?)

Then, $\triangle$ ABD and ACE are similar. (?)

In $\triangle$ ABD and ACE, sides AB and AE are homologous, as also are sides AD and AC. (§ 258)

$$\therefore AB : AD = AE : AC. \quad (?)$$

$$\therefore AB \times AC = AD \times AE \quad (?)$$
$$= AD \times (DE + AD)$$
$$= AD \times DE + \overline{AD}^2.$$

But $AD \times DE = BD \times DC$. (§ 280)

$$\therefore AB \times AC = BD \times DC + \overline{AD}^2.$$

EXERCISES.

21. The square of the altitude of an equilateral triangle is equal to three-fourths the square of the side.

22. If AD is the perpendicular from A to side BC of triangle ABC, prove

$$\overline{AB}^2 - \overline{AC}^2 = \overline{BD}^2 - \overline{CD}^2.$$

23. If one leg of a right triangle is double the other, the perpendicular from the vertex of the right angle to the hypotenuse divides it into segments which are to each other as 1 to 4. (§ 271.)

24. If two parallels to side BC of triangle ABC meet sides AB and AC at D and F, and E and G, respectively, prove

$$\frac{BD}{CE} = \frac{BF}{CG} = \frac{DF}{EG}. \quad (\S\ 247.)$$

25. C and D are respectively the middle points of a chord AB and its subtended arc. If $AD = 12$ and $CD = 8$, what is the diameter of the circle? (§ 271.)

26. If AD and BE are the perpendiculars from vertices A and B of triangle ABC to the opposite sides, prove

$$AC : DC = BC : EC.$$

(Prove $\triangle$ ACD and BCE similar.)

27. If D is the middle point of side BC of triangle ABC, right-angled at C, prove $\overline{AB}^2 - \overline{AD}^2 = 3\,\overline{CD}^2$.

28. The diameters of two concentric circles are 14 and 50 units, respectively. Find the length of a chord of the greater circle which is tangent to the smaller. (§ 273.)

29. The length of a tangent to a circle from a point 8 units distant from the nearest point of the circumference, is 12 units. What is the diameter of the circle?

(Let x represent the radius.)

30. The non-parallel sides AD and BC of trapezoid $ABCD$ intersect at O. If $AB = 15$, $CD = 24$, and the altitude of the trapezoid is 8, what is the altitude of triangle OAB? (§ 264.)

(Draw $CE \parallel AD$.)

31. If the equal sides of an isosceles right triangle are each 18 units in length, what is the length of the median drawn from the vertex of the right angle?

32. The non-parallel sides of a trapezoid are each 53 units in length, and one of the parallel sides is 56 units longer than the other. Find the altitude of the trapezoid.

33. AB is a chord of a circle, and CE is any chord drawn through the middle point C of arc AB, cutting chord AB at D. Prove AC a mean proportional between CD and CE.

(Prove ▲ ACD and ACE similar.)

34. Two secants are drawn to a circle from an outside point. If their external segments are 12 and 9, respectively, while the internal segment of the former is 8, what is the internal segment of the latter? (§ 284.)

35. If, in triangle ABC, $\angle C = 120°$, prove

$$\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2 + AC \times BC.$$

(Fig. of Prop. XXVI. $\triangle ACD$ is one-half an equilateral $\triangle$.)

36. BC is the base of an isosceles triangle ABC inscribed in a circle. If a chord AD be drawn cutting BC at E, prove

$$AD : AB = AB : AE.$$

(Prove ▲ ABD and ABE similar.)

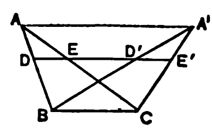
37. Two parallel chords on opposite sides of the centre of a circle are 48 units and 14 units long, respectively, and the distance between their middle points is 31 units. What is the diameter of the circle?

(Let x represent the distance from the centre to the middle point of one chord, and $31 - x$ the distance from the centre to the middle point of the other. Then the square of the radius may be expressed in two ways in terms of x.)

38. ABC is a triangle inscribed in a circle. Another circle is drawn tangent to the first externally at C, and AC and BC are produced to meet its circumference at D and E, respectively. Prove triangles ABC and CDE similar. (§ 197.)

(Draw a common tangent to the ⓢ at C. Then BC and CE are arcs of the same number of degrees.)

39. ABC and $A'BC$ are triangles whose vertices A and A' lie in a parallel to their common base BC. If a parallel to BC cuts AB and AC at D and E, and $A'B$ and $A'C$ at D' and E', respectively, prove $DE = D'E'$.

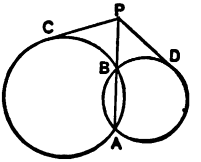


$\left(\text{Prove } \dfrac{DE}{BC} = \dfrac{D'E'}{BC}.\right)$

40. A line parallel to the bases of a trapezoid, passing through the intersection of the diagonals, and terminating in the non-parallel sides, is bisected by the diagonals. (Ex. 39.)

41. If the sides of triangle ABC are $AB = 10$, $BC = 14$, and $CA = 16$, find the lengths of the three medians. (§ 279, I.)

42. If the sides of a triangle are $AB = 4$, $AC = 5$, and $BC = 6$, find the length of the bisector of angle A. (§§ 249, 288.)



43. The tangents to two intersecting circles from any point in their common chord produced are equal. (§ 282.)

44. If two circles intersect, their common chord produced bisects their common tangents.

45. AB and AC are the tangents to a circle from point A. If CD be drawn perpendicular to radius OB at D, prove

$$AB : OB = BD : CD.$$

(Prove $\triangle$ OAB and BCD similar by § 262.)

46. ABC is a triangle inscribed in a circle. A line AD is drawn from A to any point of BC, and a chord BE is drawn, making $\angle ABE = \angle ADC$. Prove

$$AB \times AC = AD \times AE.$$

(Prove $AB : AE = AD : AC$.)

47. The radius of a circle is $22\frac{1}{2}$ units. Find the length of a chord which joins the points of contact of two tangents, each 30 units in length, drawn to the circle from a point without the circumference.

(By § 271, 2, the radius is a mean proportional between the distances from the centre to the chord and to the point without the circumference; in this way the distance from the centre to the chord can be found.)

48. If, in right triangle ABC, acute angle B is double acute angle A, prove $\overline{AC}^2 = 3\,\overline{BC}^2$. (Ex. 104, p. 71.)

49. Find the product of the segments of any chord drawn through a point 9 units from the centre of a circle whose diameter is 24 units.

50. The hypotenuse of a right triangle is 5, and the perpendicular to it from the opposite vertex is $2\frac{2}{5}$. Find the legs, and the segments into which the perpendicular divides the hypotenuse. (§ 271.)

(Let x represent one of the segments of the hypotenuse.)

51. State and prove the converse of Prop. XIII.

(Fig. of Prop. XIII. To prove $\angle BAD = \angle CAD$. Produce CA to E, making $AE = AB$.)

52. State and prove the converse of Prop. XIV.

(Fig. of Prop. XIV. Lay off $AF = AB$.)

53. If D is the middle point of hypotenuse AB of right triangle ABC, prove

$$\overline{CD}^2 = \tfrac{1}{4}(\overline{AB}^2 + \overline{BC}^2 + \overline{CA}^2). \quad \text{(Ex. 83, p. 69.)}$$

54. If a line be drawn from vertex C of isosceles triangle ABC, meeting base AB produced at D, prove

$$\overline{CD}^2 - \overline{CB}^2 = AD \times BD. \quad (\S\, 278.)$$

55. If AB is the base of isosceles triangle ABC, and AD be drawn perpendicular to BC, prove

$$3\,\overline{AD}^2 + \overline{BD}^2 + 2\,\overline{CD}^2 = \overline{AB}^2 + \overline{BC}^2 + \overline{CA}^2.$$

(We have $3\,\overline{AD}^2 = \overline{AD}^2 + 2\,\overline{AD}^2$.)

56. The middle points of two chords are distant 5 and 9 units, respectively, from the middle points of their subtended arcs. If the length of the first chord is 20 units, find the length of the second.

(Find the diameter by aid of § 270, 1.)

57. The sides AB and AC, of triangle ABC, are 16 and 9, respectively, and the length of the median drawn from C is 11. Find side BC. (§ 279, I.)

58. The diameter which bisects a chord whose length is $33\tfrac{3}{5}$ units, is 35 units in length. Find the distance from either extremity of the chord to the extremities of the diameter.

(Let x represent one segment of the diameter made by the chord.)

59. The equal angles of an isosceles triangle are each 30°, and the equal sides are each 8 units in length. What is the length of the base? (Ex. 104, p. 71.)

60. The diagonals of a trapezoid, whose bases are AD and BC, intersect at E. If $AE = 9$, $EC = 3$, and $BD = 16$, find BE and ED.

(△ AED and BEC are similar. Find BE by § 237.)

61. Prove the theorem of § 284 by drawing $A'B$ and AB'.

62. The parallel sides, AD and BC, of a circumscribed trapezoid are 18 and 6, respectively, and the other two sides are equal to each other. Find the diameter of the circle.

(Find AB by Ex. 31, p. 100. Draw through B a ∥ to CD.)

63. An angle of a triangle is acute, right, or obtuse according as the square of the opposite side is less than, equal to, or greater than, the sum of the squares of the other two sides.

(Prove by *Reductio ad Absurdum*.)

64. Is the greatest angle of a triangle whose sides are 3, 5, and 6, acute, right, or obtuse?

65. Is the greatest angle of a triangle whose sides are 8, 9, and 12, acute, right, or obtuse?

66. Is the greatest angle of a triangle whose sides are 12, 35, and 37, acute, right, or obtuse?

67. If two adjacent sides and one of the diagonals of a parallelogram are 7, 9, and 8, respectively, find the other diagonal.

(One-half of either diagonal is a median of the △ whose sides are, respectively, the given sides and the other diagonal of the ▱.)

68. If D is the intersection of the perpendiculars from the vertices of triangle ABC to the opposite sides, prove

$$\overline{AB}^2 - \overline{AC}^2 = \overline{BD}^2 - \overline{CD}^2. \quad (\S\ 272.)$$

69. If a parallel to hypotenuse AB of right triangle ABC meets AC and BC at D and E, respectively, prove

$$\overline{AE}^2 + \overline{BD}^2 = \overline{AB}^2 + \overline{DE}^2.$$

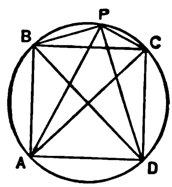
70. The diameters of two circles are 12 and 28, respectively, and the distance between their centres is 29. Find the length of the common tangent which cuts the straight line joining the centres.

(Find the ⊥ drawn from the centre of the smaller ⊙ to the radius of the greater ⊙ produced through the point of contact.)

71. State and prove the converse of Prop. XXIII., III.

(Fig. of Prop. XXIII. △s ABC and ACD are similar.)

72. State and prove the converse of Prop. XXIII., II.



73. The sum of the squares of the distances of any point in the circumference of a circle from the vertices of an inscribed square, is equal to twice the square of the diameter of the circle. (§ 195.)

(To prove $\overline{PA}^2 + \overline{PB}^2 + \overline{PC}^2 + \overline{PD}^2 = 2\,\overline{AC}^2$.)

74. The sides AB, BC, and CA, of triangle ABC, are 13, 14, and 15, respectively. Find the segments into which AB and BC are divided by perpendiculars drawn from C and A, respectively.

(∠s BAC and ACB are acute by § 98. Find the segments by § 277.)

75. In right triangle ABC is inscribed a square $DEFG$, having its vertices D and G in hypotenuse BC, and its vertices E and F in sides AB and AC, respectively. Prove $BD : DE = DE : CG$.

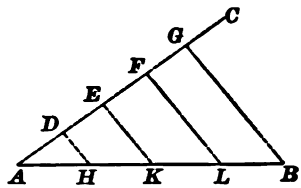
(Prove △s BDE and CFG similar.)

Note. For additional exercises on Book III., see p. 226.

CONSTRUCTIONS.

Prop. XXXII. Problem.

289. *To divide a given straight line into any number of equal parts.*



Given line AB.

Required to divide AB into four equal parts.

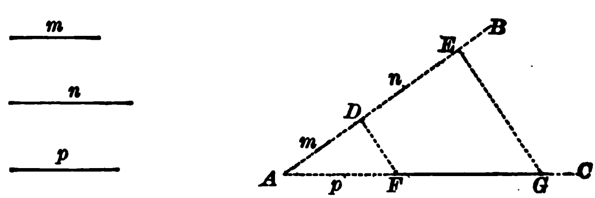
Construction. On the indefinite line AC, take any convenient length AD; on DC take $DE = AD$; on EC take $EF = AD$; on FC take $FG = AD$; and draw line BG.

Draw lines DH, EK, and $FL \parallel BG$, meeting AB at H, K, and L, respectively.

$$\therefore AH = HK = KL = LB. \qquad (\S\ 242)$$

Prop. XXXIII. Problem.

290. *To construct a fourth proportional* (§ 231) *to three given straight lines.*



Given lines m, n, and p.

Required to construct a fourth proportional to m, n, and p.

Construction. Draw the indefinite lines AB and AC, making any convenient $\angle$ with each other.

On AB take $AD = m$; on DB take $DE = n$; on AC take $AF = p$.

Draw line DF; also, line $EG \parallel DF$, meeting AC at G.

Then, FG is a fourth proportional to m, n, and p.

Proof. Since DF is $\parallel$ to side EG of $\triangle AEG$,

$$AD : DE = AF : FG. \quad (?)$$

That is, $$m : n = p : FG.$$

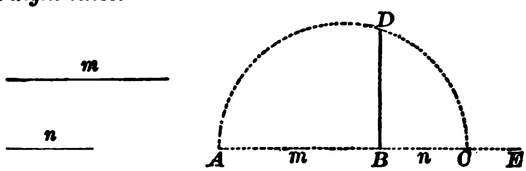
291. Cor. If we take $AF = n$, the proportion becomes

$$m : n = n : FG.$$

In this case, FG is a *third proportional* (§ 230) to m and n.

Prop. XXXIV. Problem.

292. *To construct a mean proportional* (§ 230) *between two given straight lines.*



Given lines m and n.

Required to construct a mean proportional between m and n.

Construction. On the indefinite line AE, take $AB = m$; on BE take $BC = n$.

With AC as a diameter, describe the semi-circumference ADC.

Draw line $BD \perp AC$, meeting the arc at D.

Then, BD is a mean proportional between m and n.

(The proof is left to the pupil; see § 270.)

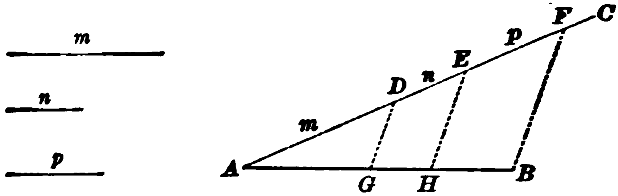
293. Sch. By aid of § 292, a line may be constructed equal to $\sqrt{a}$, where a is any number whatever.

Thus, to construct a line equal to $\sqrt{3}$, we take AB equal to 3 units, and BC equal to 1 unit.

Then, $BD = \sqrt{AB \times BC}$ (§ 232) $= \sqrt{3 \times 1} = \sqrt{3}$.

Prop. XXXV. Problem.

294. *To divide a given straight line into parts proportional to any number of given lines.*



Given line AB, and lines m, n, and p.

Required to divide AB into parts proportional to m, n, and p.

Construction. On the indefinite line AC, take $AD = m$; on DC take $DE = n$; on EC take $EF = p$; and draw line BF.

Draw lines DG and EH ∥ to BF, meeting AB at G and H, respectively.

Then, AB is divided into parts AG, GH, and HB proportional to m, n, and p, respectively.

Proof. Since DG is ∥ to side EH of $\triangle AEH$,

$$\frac{AH}{AE} = \frac{AG}{AD} = \frac{GH}{DE}. \qquad (?)$$

That is,
$$\frac{AH}{AE} = \frac{AG}{m} = \frac{GH}{n}. \qquad (1)$$

And since EH is ∥ to side BF of $\triangle ABF$,

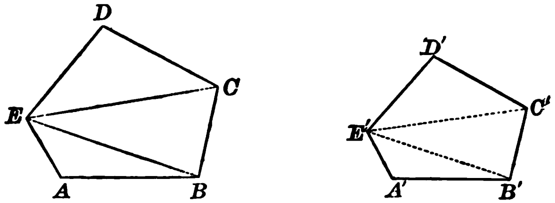
$$\frac{AH}{AE} = \frac{HB}{EF} = \frac{HB}{p}. \qquad (2)$$

From (1) and (2),
$$\frac{AG}{m} = \frac{GH}{n} = \frac{HB}{p}. \qquad (?)$$

Ex. 76. Construct a line equal to $\sqrt{2}$; to $\sqrt{5}$; to $\sqrt{6}$.

PROP. XXXVI. PROBLEM.

295. *Upon a given side, homologous to a given side of a given polygon, to construct a polygon similar to the given polygon.*



Given polygon $ABCDE$, and line $A'B'$.

Required to construct upon side $A'B'$, homologous to AB, a polygon similar to $ABCDE$.

Construction. Divide polygon $ABCDE$ into $\triangle$ by drawing diagonals EB and EC.

At A' construct $\angle B'A'E' = \angle A$; and draw line $B'E'$, making $\angle A'B'E' = \angle ABE$, meeting $A'E'$ at E'.

Then, $\triangle A'B'E'$ will be similar to $\triangle ABE$. (?)

In like manner, construct $\triangle B'C'E'$ similar to $\triangle BCE$, and $\triangle C'D'E'$ similar to $\triangle CDE$.

Then, polygon $A'B'C'D'E'$ will be similar to polygon $ABCDE$. (§ 266)

296. Def. A straight line is said to be divided by a given point in *extreme and mean ratio* when one of the segments (§ 250) is a mean proportional between the whole line and the other segment.

Thus, line AB is divided *internally* in extreme and mean ratio at C if

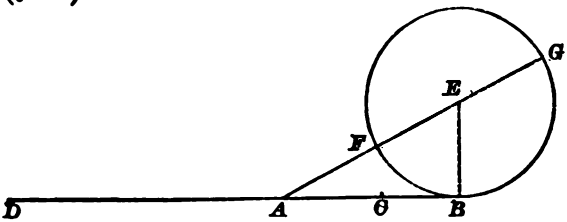
$$AB : AC = AC : BC;$$

and *externally* in extreme and mean ratio at D if

$$AB : AD = AD : BD.$$

Prop. XXXVII. Problem.

297. *To divide a given straight line in extreme and mean ratio* (§ 296).



Given line AB.

Required to divide AB in extreme and mean ratio.

Construction. Draw line $BE \perp AB$, and equal to $\frac{1}{2} AB$. With E as a centre and EB as a radius, describe $\odot BFG$. Draw line AE cutting the circumference at F and G. On AB take $AC = AF$; on BA produced, take $AD = AG$.

Then, AB is divided at C internally, and at D externally, in extreme and mean ratio.

Proof. Since AG is a secant, and AB a tangent,

$$AG : AB = AB : AF. \qquad (\S\ 283)$$

$$\therefore\ AG : AB = AB : AC. \qquad (1)$$

$$\therefore\ AG - AB : AB = AB - AC : AC. \qquad (?)$$

$$\therefore\ AB : AG - AB = AC : BC. \qquad (?)$$

But by cons., $\quad AB = 2\,BE = FG. \qquad (2)$

$$\therefore\ AG - AB = AG - FG = AF = AC.$$

Substituting, $\quad AB : AC = AC : BC. \qquad (3)$

Therefore, AB is divided at C internally in extreme and mean ratio.

Again, from (1),

$$AG + AB : AG = AB + AC : AB. \qquad (?)$$

But, $\quad AG + AB = AD + AB = BD.$

And by (2), $\quad AB + AC = FG + AF = AG.$

$$\therefore\ BD : AG = AG : AB.$$

$$\therefore\ AB : AG = AG : BD. \qquad (?)$$

$$\therefore\ AB : AD = AD : BD.$$

Therefore, AB is divided at D externally in extreme and mean ratio.

298. Cor. If AB be denoted by m, and AC by x, proportion (3) of § 297 becomes

$$m : x = x : m - x.$$

$$\therefore\ x^2 = m(m - x) = m^2 - mx. \qquad (\S\ 232)$$

Or, $$x^2 + mx = m^2.$$

Multiplying by 4, and adding m^2 to both members,

$$4x^2 + 4mx + m^2 = 4m^2 + m^2 = 5m^2.$$

Extracting the square root of both members,

$$2x + m = \pm m\sqrt{5}.$$

Since x cannot be negative, we take the positive sign before the radical sign; then,

$$2x = m\sqrt{5} - m.$$

$$\therefore\ x (\text{or } AC) = \frac{m(\sqrt{5} - 1)}{2}.$$

EXERCISES.

77. To inscribe in a given circle a triangle similar to a given triangle. (§ 261.)

(Circumscribe a ⊙ about the given △, and draw radii to the vertices.)

78. To circumscribe about a given circle a triangle similar to a given triangle. (§ 262.)

Book IV.

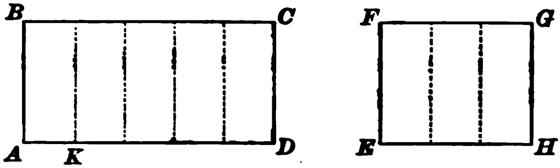
AREAS OF POLYGONS

Prop. I. Theorem.

299. *Two rectangles having equal altitudes are to each other as their bases.*

Note. The words "rectangle," "parallelogram," "triangle," etc., in the propositions of Book IV., mean the *amount of surface* in the rectangle, parallelogram, triangle, etc.

Case I. *When the bases are commensurable.*



Given rectangles $ABCD$ and $EFGH$, with equal altitudes AB and EF, and commensurable bases AD and EH.

To Prove $\frac{ABCD}{EFGH}=\frac{AD}{EH}$.

Proof. Let AK be a common measure of AD and EH, and let it be contained 5 times in AD, and 3 times in EH.

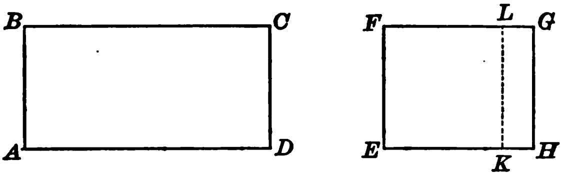
$$\therefore \frac{AD}{EH}=\frac{5}{3}. \qquad (1)$$

Drawing ⊥s to AD and EH through the several points of division, rect. $ABCD$ will be divided into 5 parts, and rect. $EFGH$ into 3 parts, all of which parts are equal. (§ 114)

$$\therefore \frac{ABCD}{EFGH}=\frac{5}{3}. \qquad (2)$$

From (1) and (2), $\frac{ABCD}{EFGH}=\frac{AD}{EH}$. (?)

Case II. *When the bases are incommensurable.*



Given rectangles $ABCD$ and $EFGH$, with equal altitudes AB and EF, and incommensurable bases AD and EH.

To Prove $$\frac{ABCD}{EFGH}=\frac{AD}{EH}.$$

Proof. Divide AD into any number of equal parts, and apply one of these parts to EH as a unit of measure.

Since AD and EH are incommensurable, a certain number of the parts will extend from E to K, leaving a remainder $KH <$ one of the equal parts.

Draw line $KL \perp EH$, meeting FG at L.

Then, since AD and EK are commensurable,

$$\frac{ABCD}{EFLK}=\frac{AD}{EK}. \qquad \text{(§ 299, Case I.)}$$

Now let the number of subdivisions of AD be indefinitely increased.

Then the unit of measure will be indefinitely diminished, and the remainder KH will approach the limit 0.

Then, $\frac{ABCD}{EFLK}$ will approach the limit $\frac{ABCD}{EFGH}$,

and $\frac{AD}{EK}$ will approach the limit $\frac{AD}{EH}$.

By the Theorem of Limits, these limits are equal. (?)

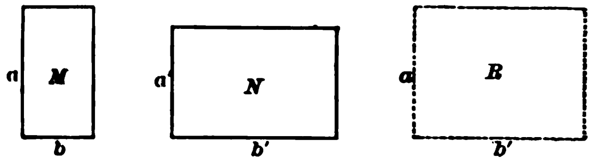
$$\therefore \frac{ABCD}{EFGH}=\frac{AD}{EH}.$$

300. Cor. Since either side of a rectangle may be taken as the base, it follows that

Two rectangles having equal bases are to each other as their altitudes.

PROP. II. THEOREM.

301. *Any two rectangles are to each other as the products of their bases by their altitudes.*



Given M and N rectangles, with altitudes a and a', and bases b and b', respectively.

To Prove $$\frac{M}{N}=\frac{a \times b}{a' \times b'}.$$

Proof. Let R be a rect. with altitude a and base b'.

Then, since rectangles M and R have equal altitudes, they are to each other as their bases. (§ 299)

$$\therefore \frac{M}{R}=\frac{b}{b'}. \qquad (1)$$

And since rectangles R and N have equal bases, they are to each other as their altitudes. (?)

$$\therefore \frac{R}{N}=\frac{a}{a'}. \qquad (2)$$

Multiplying (1) and (2), we have

$$\frac{M}{R}\times\frac{R}{N}, \text{ or } \frac{M}{N}=\frac{a \times b}{a' \times b'}.$$

DEFINITIONS.

302. The *area* of a surface is its ratio to another surface, called the *unit of surface*, adopted arbitrarily as the unit of measure (§ 184).

The usual unit of surface is a square whose side is some *linear unit*; for example, a *square inch* or a *square foot*.

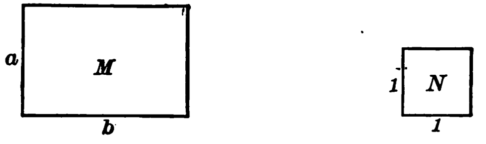
303. Two surfaces are said to be *equivalent* ($\Leftrightarrow$), when their areas are equal.

304. The *dimensions* of a rectangle are its base and altitude.

PROP. III. THEOREM.

305. *The area of a rectangle is equal to the product of its base and altitude.*

Note. In all propositions relating to areas, the unit of surface (§ 302) is understood to be *a square whose side is the linear unit.*



Given a and b, the altitude and base, respectively, of rect. M; and N the unit of surface, *i.e.*, a square whose side is the linear unit.

To Prove that, if N is the unit of surface,

$$\text{area } M = a \times b.$$

Proof. Since any two rectangles are to each other as the products of their bases by their altitudes (§ 301),

$$\frac{M}{N} = \frac{a \times b}{1 \times 1} = a \times b.$$

But since N is the unit of surface, the ratio of M to N is the *area* of M. (§ 302)

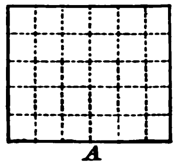
$$\therefore \text{ area } M = a \times b.$$

306. Sch. I. The statement of Prop. III. is an abbreviation of the following:

If the unit of surface is a square whose side is the linear unit, the *number* which expresses the area of a rectangle is equal to the product of the *numbers* which express the lengths of its sides.

An interpretation of this form is always understood in every proposition relating to areas.

307. Cor. *The area of a square is equal to the square of its side.*



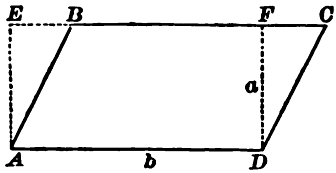
308. Sch. II. If the sides of a rectangle are *multiples* of the linear unit, the truth of Prop. III. may be seen by dividing the figure into squares, each equal to the unit of surface.

Thus, if the altitude of rectangle A is 5 units, and its base 6 units, the figure can be divided into 30 squares.

In this case, 30, the number which expresses the area of the rectangle, is the product of 6 and 5, the numbers which express the lengths of the sides.

Prop. IV. Theorem.

309. *The area of a parallelogram is equal to the product of its base and altitude.*



Given ▱ $ABCD$, with its altitude $DF = a$, and its base $AD = b$.

To Prove area $ABCD = a \times b$.

Proof. Draw line $AE \parallel DF$, meeting CB produced at E.

Then, $AEFD$ is a rectangle. (?)

In rt. ▵ ABE and DCF,

$$AB = DC, \text{ and } AE = DF. \quad (?)$$

$$\therefore \triangle ABE = \triangle DCF. \quad (?)$$

Now if from the entire figure $ADCE$ we take $\triangle ABE$, there remains ▱ $ABCD$; and if we take $\triangle DCF$, there remains rect. $AEFD$.

$$\therefore \text{ area } ABCD = \text{area } AEFD = a \times b. \quad (\S\ 305)$$

310. Cor. I. *Two parallelograms having equal bases and equal altitudes are equivalent* (§ 303).

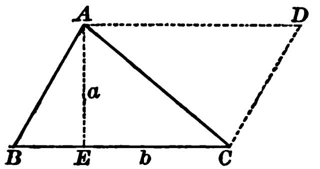
311. Cor. II. 1. *Two parallelograms having equal altitudes are to each other as their bases.*

2. *Two parallelograms having equal bases are to each other as their altitudes.*

3. *Any two parallelograms are to each other as the products of their bases by their altitudes.*

PROP. V. THEOREM.

312. *The area of a triangle is equal to one-half the product of its base and altitude.*



Given $\triangle ABC$, with its altitude $AE = a$, and its base $BC = b$.

To Prove area $ABC = \frac{1}{2} a \times b$.

(By § 108, AC divides $\square ABCD$ into two equal $\triangle$s.)

313. Cor. I. *Two triangles having equal bases and equal altitudes are equivalent.*

314. Cor. II. 1. *Two triangles having equal altitudes are to each other as their bases.*

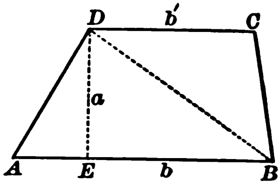
2. *Two triangles having equal bases are to each other as their altitudes.*

3. *Any two triangles are to each other as the products of their bases by their altitudes.*

315. Cor. III. *A triangle is equivalent to one-half of a parallelogram having the same base and altitude.*

PROP. VI. THEOREM.

316. *The area of a trapezoid is equal to one-half the sum of its bases multiplied by its altitude.*



Given trapezoid $ABCD$, with its altitude DE equal to a, and its bases AB and DC equal to b and b', respectively.

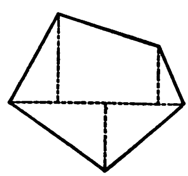
To Prove area $ABCD = a \times \frac{1}{2}(b + b')$.

(The trapezoid is composed of two $\triangle$ whose altitude is a, and bases b and b', respectively.)

317. Cor. Since the line joining the middle points of the non-parallel sides of a trapezoid is equal to one-half the sum of the bases (§ 132), it follows that

The area of a trapezoid is equal to the product of its altitude by the line joining the middle points of its non-parallel sides.

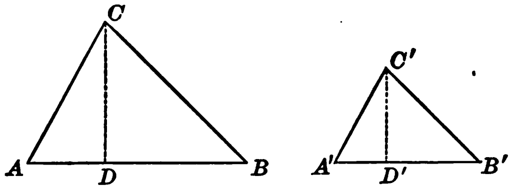
318. Sch. The area of any polygon may be obtained by finding the sum of the areas of the triangles into which the polygon may be divided by drawing diagonals from any one of its vertices.



But in practice it is better to draw the longest diagonal, and draw perpendiculars to it from the remaining vertices of the polygon. The polygon will then be divided into right triangles and trapezoids; and by measuring the lengths of the perpendiculars, and of the portions of the diagonal which they intercept, the areas of the figures may be found by §§ 312 and 316.

Prop. VII. Theorem.

319. *Two similar triangles are to each other as the squares of their homologous sides.*



Given AB and $A'B'$ homologous sides of similar $\triangle ABC$ and $A'B'C'$, respectively.

To Prove $$\frac{ABC}{A'B'C'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}.$$

Proof. Draw altitudes CD and $C'D'$.

$$\therefore \frac{ABC}{A'B'C'} = \frac{AB \times CD}{A'B' \times C'D'} \qquad (\S\ 314,\ 3)$$

$$= \frac{AB}{A'B'} \times \frac{CD}{C'D'}. \qquad (1)$$

But, $$\frac{CD}{C'D'} = \frac{AB}{A'B'}. \qquad (\S\ 264)$$

Substituting this value in (1),

$$\frac{ABC}{A'B'C'} = \frac{AB}{A'B'} \times \frac{AB}{A'B'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}.$$

320. Sch. Two similar triangles are to each other as the squares of any two homologous lines.

EXERCISES.

1. If the area of a rectangle is 7956 sq. in., and its base $3\frac{1}{4}$ yd., find its perimeter in feet.

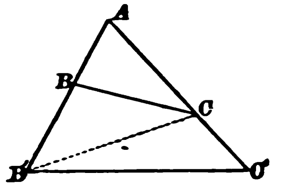
2. If the base and altitude of a rectangle are 14 ft. 7 in., and 5 ft. 3 in., respectively, what is the side of an equivalent square?

3. Find the dimensions of a rectangle whose area is 168, and perimeter 52.

(Let x represent the base.)

PROP. VIII. THEOREM.

321. ***Two triangles having an angle of one equal to an angle of the other, are to each other as the products of the sides including the equal angles.***



Given $\angle A$ common to $\triangle ABC$ and $AB'C'$.

To Prove $$\frac{ABC}{AB'C'} = \frac{AB \times AC}{AB' \times AC'}.$$

Proof. Draw line $B'C$.

Then $\triangle ABC$ and $AB'C$, having the common vertex C, and their bases AB and AB' in the same str. line, have the same altitude.

$$\therefore \frac{ABC}{AB'C} = \frac{AB}{AB'} \qquad (\S\ 314, 1)$$

And $\triangle AB'C$ and $AB'C'$, having the common vertex B', and their bases AC and AC' in the same str. line, have the same altitude.

$$\therefore \frac{AB'C}{AB'C'} = \frac{AC}{AC'}.$$

Multiplying these equations, we have

$$\frac{ABC}{AB'C} \times \frac{AB'C}{AB'C'}, \text{ or } \frac{ABC}{AB'C'} = \frac{AB \times AC}{AB' \times AC'}.$$

EXERCISES.

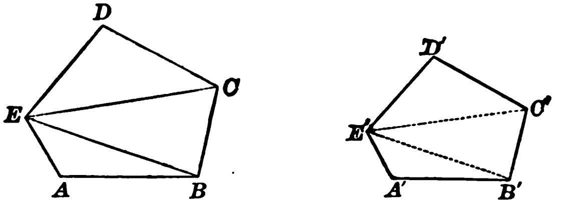
4. The area of a rectangle is 143 sq. ft. 75 sq. in., and its base is 3 times its altitude. Find each of its dimensions.

(Let x represent the altitude.)

5. The hypotenuse of a right triangle is 5 ft. 5 in., and one of its legs is 2 ft. 9 in. Find its area.

PROP. IX. THEOREM.

322. ***Two similar polygons are to each other as the squares of their homologous sides.***



Given AB and $A'B'$ homologous sides of similar polygons AC and $A'C'$, whose areas are K and K', respectively.

To Prove $$\frac{K}{K'}=\frac{\overline{AB}^2}{\overline{A'B'}^2}.$$

Proof. Draw diagonals EB, EC, $E'B'$, and $E'C'$.

Then, $\triangle ABE$ is similar to $\triangle A'B'E'$. (§ 267)

$$\therefore \frac{ABE}{A'B'E'}=\frac{\overline{AB}^2}{\overline{A'B'}^2}. \quad (§ 319)$$

In like manner,

$$\frac{BCE}{B'C'E'}=\frac{\overline{BC}^2}{\overline{B'C'}^2}=\frac{\overline{AB}^2}{\overline{A'B'}^2},$$

and $$\frac{CDE}{C'D'E'}=\frac{\overline{CD}^2}{\overline{C'D'}^2}=\frac{\overline{AB}^2}{\overline{A'B'}^2}. \quad (§ 253, 2)$$

$$\therefore \frac{ABE}{A'B'E'}=\frac{BCE}{B'C'E'}=\frac{CDE}{C'D'E'}. \quad (?)$$

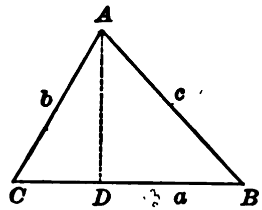
$$\therefore \frac{ABE+BCE+CDE}{A'B'E'+B'C'E'+C'D'E'}=\frac{ABE}{A'B'E'}. \quad (§ 240)$$

$$\therefore \frac{K}{K'}=\frac{ABE}{A'B'E'}=\frac{\overline{AB}^2}{\overline{A'B'}^2}.$$

323. Cor. *Two similar polygons are to each other as the squares of their perimeters.* (§ 268)

PROP. X. PROBLEM.

324. *To express the area of a triangle in terms of its three sides.*



Given sides BC, CA, and AB, of $\triangle ABC$, equal to a, b, and c, respectively.

Required to express area ABC in terms of a, b, and c.

Solution. Let C be an acute $\angle$, and draw altitude AD.

$$\therefore\ c^2 = a^2 + b^2 - 2a \times CD. \quad (\S\ 277)$$

Transposing, $2a \times CD = a^2 + b^2 - c^2$.

$$\therefore\ CD = \frac{a^2 + b^2 - c^2}{2a}.$$

$$\therefore\ \overline{AD}^2 = \overline{AC}^2 - \overline{CD}^2 \quad (\S\ 273)$$

$$= (AC + CD)(AC - CD)$$

$$= \left(b + \frac{a^2 + b^2 - c^2}{2a}\right)\left(b - \frac{a^2 + b^2 - c^2}{2a}\right)$$

$$= \frac{(2ab + a^2 + b^2 - c^2)(2ab - a^2 - b^2 + c^2)}{4a^2}$$

$$= \frac{[(a + b)^2 - c^2][c^2 - (a - b)^2]}{4a^2}$$

$$= \frac{(a + b + c)(a + b - c)(c + a - b)(c - a + b)}{4a^2}. \quad (1)$$

Now let $a + b + c = 2s$.

$$\therefore\ \overline{AD}^2 = \frac{2s(2s - 2c)(2s - 2b)(2s - 2a)}{4a^2}$$

$$= \frac{16s(s - a)(s - b)(s - c)}{4a^2}.$$

$$\therefore AD = \frac{2\sqrt{s(s-a)(s-b)(s-c)}}{a}.$$

$$\therefore \text{area } ABC = \tfrac{1}{2} a \times AD \qquad (?)$$

$$= \sqrt{s(s-a)(s-b)(s-c)}.$$

325. Sch. Let it be required to find the area of a triangle whose sides are 13, 14, and 15.

Let $a = 13$, $b = 14$, and $c = 15$; then

$$s = \tfrac{1}{2}(13 + 14 + 15) = 21.$$

Whence, $s - a = 8$, $s - b = 7$, and $s - c = 6$.

Then, the area of the triangle is

$$\sqrt{21 \times 8 \times 7 \times 6} = \sqrt{3 \times 7 \times 2^3 \times 7 \times 2 \times 3}$$

$$= \sqrt{2^4 \times 3^2 \times 7^2} = 2^2 \times 3 \times 7 = 84.$$

EXERCISES.

6. Find the area of a triangle whose sides are 8, 13, and 15.

7. The area of a square is 693 sq. yd. 4 sq. ft.; find its side.

8. If the altitude of a trapezoid is 1 ft. 4 in., and its bases 1 ft. 1 in. and 2 ft. 5 in., respectively, what is its area?

9. If, in figure of Prop. VII., $AB = 9$, $A'B' = 7$, and the area of $A'B'C'$ is 147, find area ABC.

10. If the sides of triangle ABC are $AB = 25$, $BC = 17$, and $CA = 28$, find its area, and the length of the perpendicular from each vertex to the opposite side.

11. Find the length of the diagonal of a rectangle whose area is 2640, and altitude 48.

12. Find the lower base of a trapezoid whose area is 9408, upper base 79, and altitude 96.

13. The area of a rhombus is equal to one-half the product of its diagonals. (§ 117.)

14. The diagonals of a parallelogram divide it into four equivalent triangles.

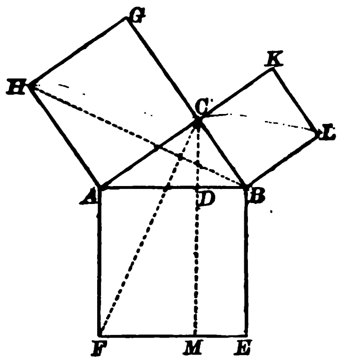
15. Lines drawn to the vertices of a parallelogram from any point in one of its diagonals divide the figure into two pairs of equivalent triangles. (Ex. 63, p. 67.)

16. The area of a certain triangle is $2\frac{1}{4}$ times the area of a similar triangle. If the altitude of the first triangle is 4 ft. 3 in., what is the homologous altitude of the second? (§ 320.)

326. Sch. Since the area of a square is equal to the square of its side (§ 307), we may state Prop. XXIV., Book III., as follows:

In any right triangle, the square described upon the hypotenuse is equivalent to the sum of the squares described upon the legs.

The theorem in the above form may be proved as follows:



Given $ABEF$, $ACGH$, and $BCKL$ squares described upon hypotenuse AB, and legs AC and BC, respectively, of rt. $\triangle ABC$.

To Prove area $ABEF =$ area $ACGH +$ area $BCKL$.

Proof. Draw line $CD \perp AB$, and produce it to meet EF at M; also, draw lines BH and CF.

Then in $\triangle$ ABH and ACF, by hyp.,

$$AB = AF \text{ and } AH = AC.$$

Also, $\angle BAH = \angle CAF$,

for each is equal to a rt. $\angle + \angle BAC$.

$$\therefore \triangle ABH = \triangle ACF. \qquad (?)$$

Now $\triangle ABH$ has the same base and altitude as square $ACGH$.

$$\therefore \text{area } ABH = \tfrac{1}{2} \text{ area } ACGH. \qquad (\S\ 315)$$

And $\triangle ACF$ has the same base and altitude as rect. $ADMF$.

$$\therefore \text{ area } ACF = \tfrac{1}{2} \text{ area } ADMF.$$

But, $$\text{area } ABH = \text{area } ACF.$$

$$\therefore \tfrac{1}{2} \text{ area } ACGH = \tfrac{1}{2} \text{ area } ADMF, \qquad (?)$$

or $$\text{area } ACGH = \text{area } ADMF. \qquad (1)$$

Similarly, by drawing lines AL and CE, we may prove

$$\text{area } BCKL = \text{area } BDME. \qquad (2)$$

Adding (1) and (2), we have

$$\text{area } ACGH + \text{area } BCKL = \text{area } ABEF.$$

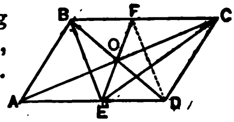
327. Sch. The theorem of § 326 is supposed to have been first given by Pythagoras, and is called after him the *Pythagorean Theorem.*

Several other propositions of Book III. may be put in the form of statements in regard to areas; as, for example, Props. XXV. and XXVI.

EXERCISES.

17. If EF is any straight line drawn through the centre of parallelogram $ABCD$, meeting sides AD and BC at E and F, respectively, prove triangles BEF and CED equivalent. (Ex. 61, p. 66.)

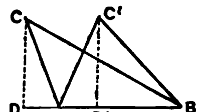
(Prove $BEDF$ a ▱ by § 112.)



18. The side of an equilateral triangle is 5; find its area. (Ex. 21, p. 151.)

19. The altitude of an equilateral triangle is 3; find its area.

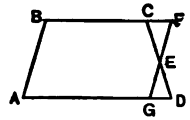
20. Two triangles are equivalent if they have two sides of one equal respectively to two sides of the other, and the included angles supplementary.



21. One diagonal of a rhombus is five-thirds of the other, and the difference of the diagonals is 8; find its area. (Ex. 13, p. 173.)

22. If D and E are the middle points of sides BC and AC, respectively, of triangle ABC, prove triangles ABD and ABE equivalent. (§ 80.)

23. If E is the middle point of CD, one of the non-parallel sides of trapezoid $ABCD$, and a parallel to AB drawn through E meets BC at F and AD at G, prove parallelogram $ABFG$ equivalent to the trapezoid.



24. The sides AB, BC, CD, and DA of quadrilateral $ABCD$ are 10, 17, 13, and 20, respectively, and the diagonal AC is 21. Find the area of the quadrilateral.

25. Find the area of the square inscribed in a circle whose radius is 3.

(The diagonal is a diameter, by § 157.)

26. The area of an isosceles right triangle is 81 sq. in.; find its hypotenuse in feet.

(Represent one of the equal sides by x.)

27. The area of an equilateral triangle is $9\sqrt{3}$; find its side.

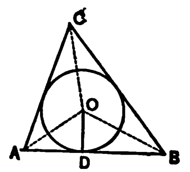
(Represent the side by x.)

28. The area of an equilateral triangle is $16\sqrt{3}$; find its altitude.

(Represent the altitude by x.)

29. The base of an isosceles triangle is 56, and each of the equal sides is 53; find its area.

30. The area of a triangle is equal to one-half the product of its perimeter by the radius of the inscribed circle.



31. The area of an isosceles right triangle is equal to one-fourth the area of the square described upon the base. (§ 307.)

32. If angle A of triangle ABC is 30°, prove

$$\text{area } ABC = \tfrac{1}{4} AB \times AC.$$

(Draw $CD \perp AB$; then CD may be found by Ex. 104, p. 71.)

33. A circle whose diameter is 12 is inscribed in a quadrilateral whose perimeter is 50. Find the area of the quadrilateral.

(Compare Ex. 30, p. 176.)

34. Two similar triangles have homologous sides equal to 8 and 15, respectively. Find the homologous side of a similar triangle equivalent to their sum. (§ 319.)

35. If E is any point within parallelogram $ABCD$, triangles ABE and CDE are together equivalent to one-half the parallelogram.

(Draw through E a $\parallel$ to AB.)

36. The non-parallel sides, AB and CD, of a trapezoid are each 25 units in length, and the sides AD and BC are 33 and 19 units, respectively. Find the area of the trapezoid.

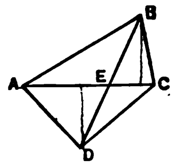
(Draw through B a $\parallel$ to CD, and a $\perp$ to AD.)

37. If the area of a polygon, one of whose sides is 15 in., is 375 sq. in., what is the area of a similar polygon whose homologous side is 18 in.?

38. If the area of a polygon, one of whose sides is 36 ft., is 648 sq. ft., what is the homologous side of a similar polygon whose area is 392 sq. ft.?

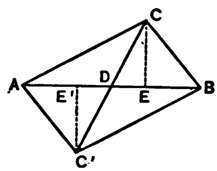
39. If one diagonal of a quadrilateral bisects the other, it divides the quadrilateral into two equivalent triangles.

(To prove $\triangle ABC \Leftrightarrow \triangle ACD$.)



40. Two equivalent triangles have a common base, and lie on opposite sides of it. Prove that the base, produced if necessary, bisects the line joining their vertices.

(To prove $CD = C'D$.)

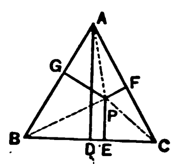


41. If the sides of a triangle are 15, 41, and 52, find the radius of the inscribed circle. (Ex. 30, p. 176.)

42. The area of a rhombus is 240, and its side is 17; find its diagonals. (Ex. 13, p. 173.)

(Represent the diagonals by $2x$ and $2y$.)

43. The sum of the perpendiculars from any point within an equilateral triangle to the three sides is equal to the altitude of the triangle.



44. The longest sides of two similar polygons are 18 and 3, respectively. How many polygons, each equal to the second, will form a polygon equivalent to the first? (§ 322.)

45. If the sides of a triangle are 25, 29, and 36, find the diameter of the circumscribed circle. (§ 287.)

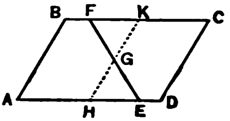
(The altitude of a $\triangle$ equals its area divided by one-half its base.)

46. If a is the base, and b one of the equal sides of an isosceles triangle, prove its area equal to $\frac{1}{4}a\sqrt{4b^2 - a^2}$.

47. The sides AB and AC of triangle ABC are 15 and 22, respectively. From a point D in AB, a parallel to BC is drawn meeting AC at E, and dividing the triangle into two equivalent parts. Find AD and AE. (§ 319.)

48. The segments of the hypotenuse of a right triangle made by a perpendicular drawn from the vertex of the right angle, are $5\frac{3}{5}$ and $9\frac{2}{5}$, respectively; find the area of the triangle.

49. Any straight line drawn through the centre of a parallelogram, terminating in a pair of opposite sides, divides the parallelogram into two equivalent quadrilaterals. (Ex. 61, p. 66.)



50. If E is the middle point of CD, one of the non-parallel sides of trapezoid $ABCD$, prove triangle ABE equivalent to $\frac{1}{2}ABCD$.

(Draw through E a ‖ to AB.)

51. The sides of triangle ABC are $AB = 13$, $BC = 14$, and $CA = 15$. If AD is the bisector of angle A, meeting BC at D, find the areas of triangles ABD and ACD. (§§ 249, 325.)

52. The longest diagonal AD of pentagon $ABCDE$ is 44, and the perpendiculars to it from B, C, and E are 24, 16, and 15, respectively. If $AB = 25$, $CD = 20$, and $AE = 17$, what is the area of the pentagon? (§ 318.)

53. The sides of a triangle are proportional to the numbers 7, 24, and 25, respectively. The perpendicular to the third side from the vertex of the opposite angle is $13\frac{11}{25}$. Find the area of the triangle.

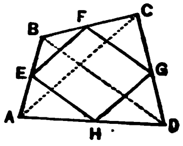
(Represent the sides by $7x$, $24x$, and $25x$, respectively; the $\triangle$ is a rt. $\triangle$ by Ex. 63, p. 154.)

54. If E and F are the middle points of sides AB and AC, respectively, of a triangle, and D is any point in BC, prove quadrilateral $AEDF$ equivalent to one-half triangle ABC.

(Prove $\triangle DEF \approx \frac{1}{2}\triangle ABC$, by aid of Ex. 64, p. 67.)

55. If E, F, G, and H are the middle points of sides AB, BC, CD, and DA, respectively, of quadrilateral $ABCD$, prove $EFGH$ a parallelogram equivalent to one-half $ABCD$.

(By Ex. 64, p. 67, area $EBF = \frac{1}{4}$ area ABC.)

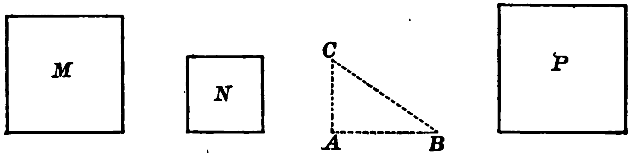


Note. For additional exercises on Book IV., see p. 229.

CONSTRUCTIONS.

PROP. XI. PROBLEM.

328. *To construct a square equivalent to the sum of two given squares.*



Given squares M and N.

Required to construct a square $\simeq M + N$.

Construction. Draw line AB equal to a side of M.

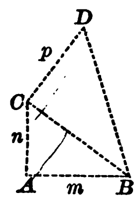
At A draw line $AC \perp AB$, and equal to a side of N; and draw line BC.

Then, square P, described with its side equal to BC, will be $\simeq M + N$.

Proof. In rt. $\triangle ABC$, $\overline{BC}^2 = \overline{AB}^2 + \overline{AC}^2$. (?)

$\therefore$ area $P =$ area $M +$ area N. (§ 307)

329. Cor. By an extension of the above method, a square may be constructed equivalent to the sum of any number of given squares.



Given three squares whose sides are equal to m, n, and p, respectively.

Required to construct a square $\simeq$ the sum of the given squares.

Construction. Draw line $AB = m$.

Draw line $AC \perp AB$, and equal to n, and line BC.

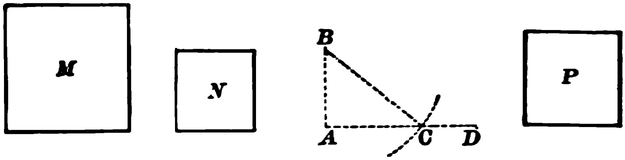
Draw line $CD \perp BC$, and equal to p, and line BD.

Then, the square described with its side equal to BD will be $\simeq$ the sum of the given squares.

(The proof is left to the pupil.)

PROP. XII. PROBLEM.

330. *To construct a square equivalent to the difference of two given squares.*



Given squares M and N, M being $> N$.

Required to construct a square $\approx M - N$.

Proof. Draw the indefinite line AD.

At A draw line $AB \perp AD$, and equal to a side of N.

With B as a centre, and with a radius equal to a side of M, describe an arc cutting AD at C, and draw line BC.

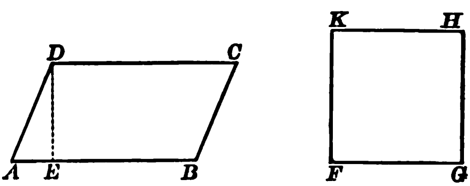
Then, square P, described with its side equal to AC, will be $\approx M - N$.

Proof. In rt. $\triangle ABC$, $\overline{AC}^2 = \overline{BC}^2 - \overline{AB}^2$. (?)

$\therefore$ area $P =$ area $M -$ area N. (?)

PROP. XIII. PROBLEM.

331. *To construct a square equivalent to a given parallelogram.*



Given $\square ABCD$.

Required to construct a square $\approx ABCD$.

Construction. Draw line $DE \perp AB$, and construct line FG a mean proportional between lines AB and DE (§ 292).

Then, square $FGHK$, described with its side equal to FG, will be $\approx ABCD$.

Proof. By cons., $AB : FG = FG : DE.$

$$\therefore \overline{FG}^2 = AB \times DE. \qquad (?)$$

$$\therefore \text{area } FGHK = \text{area } ABCD. \qquad (?)$$

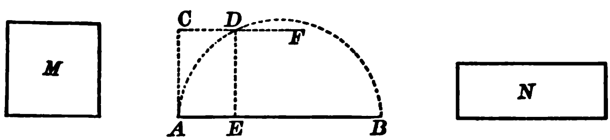
332. Cor. *A square may be constructed equivalent to a given triangle by taking for its side a mean proportional between the base and one-half the altitude of the triangle.*

Ex. 56. To construct a triangle equivalent to a given square, having given its base and an angle adjacent to the base.

(Take for the required altitude a third proportional to one-half the given base and the side of the given square.)

PROP. XIV. PROBLEM.

333. *To construct a rectangle equivalent to a given square, having the sum of its base and altitude equal to a given line.*



Given square M, and line AB.

Required to construct a rectangle $\approx M$, having the sum of its base and altitude equal to AB.

Construction. With AB as a diameter, describe semicircumference ADB.

Draw line $AC \perp AB$, and equal to a side of M.

Draw line $CF \parallel AB$, intersecting arc ADB at D, and line $DE \perp AB$.

Then, rectangle N, constructed with its base and altitude equal to BE and AE, respectively, will be $\approx M$.

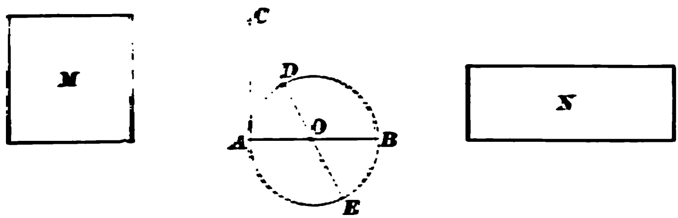
Proof. $AE : DE = DE : BE.$ (§ 270, 1)

$$\therefore AE \times BE = \overline{DE}^2 = \overline{AC}^2. \qquad (?)$$

$$\therefore \text{area } N = \text{area } M. \qquad (?)$$

PROP. XV. PROBLEM.

334. *To construct a rectangle equivalent to a given square, having the difference of its base and altitude equal to a given line.*



Given square M, and line AB.

Required to construct a rectangle $\approx M$, having the difference of its base and altitude equal to AB.

Construction. With AB as a diameter, describe $\odot ADB$.

Draw line $AC \perp AB$, and equal to a side of M.

Through centre O draw line CO, intersecting the circumference at D and E.

Then, rectangle N, constructed with its base and altitude equal to CE and CD, respectively, will be $\approx M$.

Proof. $\qquad CE - CD = DE = AB. \qquad$ (?)

That is, the difference of the base and altitude of N is equal to AB.

Again, AC is tangent to $\odot ADB$ at A. (?)

$\therefore CD \times CE = \overline{CA}^2$. (§ 282)

$\therefore$ area $N =$ area M. (?)

EXERCISES.

57. To construct a triangle equivalent to a given triangle, having given its base.

(Take for the required altitude a fourth proportional to the given base, and the base and altitude of the given $\triangle$.)

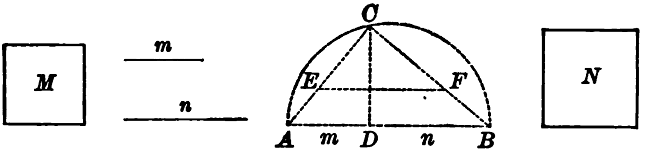
How many different $\triangle$ can be constructed?

58. To construct a rectangle equivalent to a given rectangle, having given its base.

59. To construct a square equivalent to twice a given square. (§ 307.)

PROP. XVI. PROBLEM.

335. *To construct a square having a given ratio to a given square.*



Given square M, and lines m and n.

Required to construct a square having to M the ratio $n : m$.

Construction. On line AB, take $AD = m$ and $DB = n$.

With AB as a diameter, describe semi-circumference ACB.

Draw line $DC \perp AB$, meeting arc ACB at C, and lines AC and BC.

On AC take CE equal to a side of M; and draw line $EF \parallel AB$, meeting BC at F.

Then, square N, constructed with its side equal to CF, will have to M the ratio $n : m$.

Proof. $\angle ACB$ is a rt. $\angle$. (?)

Then since CD is $\perp AB$,

$$\frac{\overline{AC}^2}{\overline{BC}^2} = \frac{AB \times AD}{AB \times BD} = \frac{AD}{BD} = \frac{m}{n}. \qquad (\S\ 271,\ 2)$$

But since EF is $\parallel AB$,

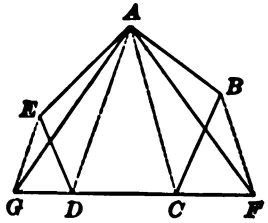
$$\frac{CE}{CF} = \frac{AC}{BC}. \qquad (?)$$

$$\therefore \frac{\overline{CE}^2}{\overline{CF}^2} = \frac{\overline{AC}^2}{\overline{BC}^2} = \frac{m}{n}.$$

$$\therefore \frac{\text{area } M}{\text{area } N} = \frac{m}{n}. \qquad (?)$$

PROP. XVII. PROBLEM.

336. *To construct a triangle equivalent to a given polygon.*



Given polygon $ABCDE$.

Required to construct a $\triangle \approx ABCDE$.

Construction. Take any three consecutive vertices, as A, B, and C, and draw diagonal AC; also, line $BF \parallel AC$, meeting DC produced at F, and line AF.

Then, $AFDE$ is a polygon $\approx ABCDE$, having a number of sides less by one.

Again, draw diagonal AD; also, line $EG \parallel AD$, meeting CD produced at G, and line AG.

Then, AFG is a $\triangle \approx ABCDE$.

Proof. $\triangle$ ABC and AFC have the same base AC.

And since their vertices B and F lie in the same line $\parallel$ to AC, they have the same altitude. (§ 80)

$\therefore$ area $ABC =$ area AFC. (?)

Adding area $ACDE$ to both members, we have

area $ABCDE =$ area $AFDE$.

Again, $\triangle$ AED and AGD have the same base AD, and the same altitude.

$\therefore$ area $AED =$ area AGD. (?)

Adding area AFD to both members, we have

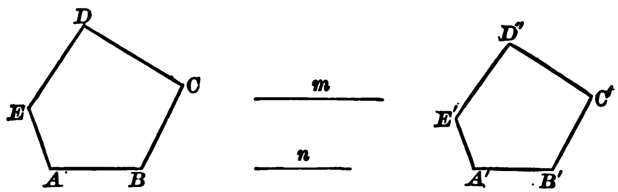
area $AFDE =$ area AFG.

$\therefore$ area $ABCDE =$ area AFG. (?)

Note. By aid of §§ 336 and 332, a square may be constructed equivalent to a given polygon.

Prop. XVIII. Problem.

337. *To construct a polygon similar to a given polygon, and having a given ratio to it.*



Given polygon AC, and lines m and n.

Required to construct a polygon similar to AC, and having to it the ratio $n : m$.

Construction. Construct $A'B'$ the side of a square having to the square described upon AB the ratio $n : m$. (§ 335)

Upon side $A'B'$, homologous to AB, construct polygon $A'C'$ similar to polygon AC. (§ 295)

Then, $A'C'$ will have to AC the ratio $n : m$.

Proof. Since AC is similar to $A'C'$,

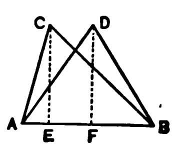
$$\frac{AC}{A'C'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}. \qquad (§ 322)$$

But by cons.,

$$\frac{\overline{AB}^2}{\overline{A'B'}^2} = \frac{m}{n}.$$

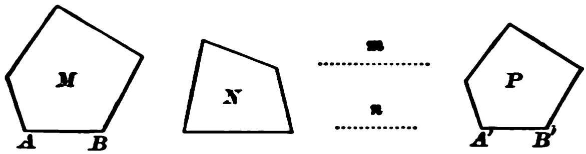
$$\therefore \frac{AC}{A'C'} = \frac{m}{n}. \qquad (?)$$

Ex. 60. To construct an isosceles triangle equivalent to a given triangle, having its base coincident with a side of the given triangle.



PROP. XIX. PROBLEM.

338. *To construct a polygon similar to one of two given polygons, and equivalent to the other.*



Given polygons M and N.

Required to construct a polygon similar to M, and $\approx N$.

Construction. Let AB be any side of M.

Construct m, the side of a square $\approx M$, and n, the side of a square $\approx N$. (Note, p. 185)

Construct $A'B'$, a fourth proportional to m, n, and AB.

Upon side $A'B'$, homologous to AB, construct polygon P similar to M. (§ 295)

Then, $P \approx N$.

Proof. Since M is similar to P,

$$\frac{\text{area } M}{\text{area } P} = \frac{\overline{AB}^2}{\overline{A'B'}^2}. \qquad (?)$$

But by cons., $m : n = AB : A'B'$, or $\dfrac{AB}{A'B'} = \dfrac{m}{n}$.

$$\therefore \frac{\text{area } M}{\text{area } P} = \frac{m^2}{n^2} = \frac{\text{area } M}{\text{area } N}. \qquad (?)$$

$$\therefore \text{area } P = \text{area } N.$$

X L

EXERCISES.

61. To construct a triangle equivalent to a given square, having given its base and the median drawn from the vertex to the base.

(Draw a ∥ to the base at a distance equal to the altitude of the △.)

What restriction is there on the values of the given lines?

62. To construct a rhombus equivalent to a given parallelogram, having one of its diagonals coincident with a diagonal of the parallelogram. (Ex. 60.)

63. To draw through a given point within a parallelogram a straight line dividing it into two equivalent parts. (Ex. 49, p. 178.)

64. To construct a parallelogram equivalent to a given trapezoid, having a side and two adjacent angles coincident with one of the non-parallel sides and the adjacent angles, respectively, of the trapezoid. (Ex. 23, p. 176.)

65. To construct a triangle equivalent to a given triangle, having given two of its sides. (Ex. 57.)

(Let m and n be the given sides, and take m as the base.)

Discuss the solution when the altitude is $< n$. $= n$. $> n$.

66. To construct a right triangle equivalent to a given square, having given its hypotenuse. (Ex. 96, p. 119.)

(Find the altitude as in Ex. 56.)

What restriction is there on the values of the given parts?

67. To construct a right triangle equivalent to a given triangle, having given its hypotenuse.

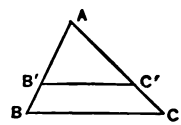
What restriction is there on the values of the given parts?

68. To construct an isosceles triangle equivalent to a given triangle, having given one of its equal sides equal to m.

(Draw a ∥ to the given side at a distance equal to the altitude.)

Discuss the solution when the altitude is $< m$. $= m$. $> m$.

69. To draw a line parallel to the base of a triangle dividing it into two equivalent parts. (§ 319.)



(△ ABC and $AB'C'$ are similar.)

70. To draw through a given point in a side of a parallelogram a straight line dividing it into two equivalent parts.

71. To draw a straight line perpendicular to the bases of a trapezoid, dividing the trapezoid into two equivalent parts.

(A str. line connecting the middle points of the bases divides the trapezoid into two equivalent parts.)

72. To draw through a given point in one of the bases of a trapezoid a straight line dividing the trapezoid into two equivalent parts.

(A str. line connecting the middle points of the bases divides the trapezoid into two equivalent parts.)

73. To construct a triangle similar to two given similar triangles, and equivalent to their sum.

(Construct squares equivalent to the △.)

74. To construct a triangle similar to two given similar triangles, and equivalent to their difference.

Book V.

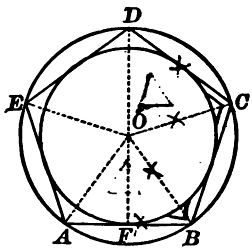
REGULAR POLYGONS.—MEASUREMENT OF THE CIRCLE.

X

339. Def. A *regular polygon* is a polygon which is both equilateral and equiangular.

Prop. I. Theorem.

340. *A circle can be circumscribed about, or inscribed in, any regular polygon.*



Given regular polygon $ABCDE$.

To Prove that a $\odot$ can be circumscribed about, or inscribed in, $ABCDE$.

Proof. Let O be the centre of the circumference described through vertices A, B, and C (§ 223).

Draw radii OA, OB, OC, and OD.

In $\triangle$ OAB and OCD, $OB = OC$. (?)

And since, by def., polygon $ABCDE$ is equilateral,

$$AB = CD.$$

Again, since, by def., polygon $ABCDE$ is equiangular,

$$\angle ABC = \angle BCD.$$

And since $\triangle OBC$ is isosceles,

$$\angle OBC = \angle OCB. \qquad (?)$$

$$\therefore \angle ABC - \angle OBC = \angle BCD - \angle OCB.$$

Or, $$\angle OBA = \angle OCD.$$

$$\therefore \triangle OAB = \triangle OCD. \qquad (?)$$

$$\therefore OA = OD. \qquad (?)$$

Then, the circumference which passes through A, B, and C also passes through D.

In like manner, it may be proved that the circumference which passes through B, C, and D also passes through E.

Hence, a $\odot$ can be circumscribed about $ABCDE$.

Again, since AB, BC, CD, etc., are equal chords of the circumscribed $\odot$, they are equally distant from O. (§ 164)

Hence, a $\odot$ described with O as a centre, and a line OF $\perp$ to any side AB as a radius, will be inscribed in $ABCDE$.

341. Def. The *centre* of a regular polygon is the common centre of the circumscribed and inscribed circles.

The *angle at the centre* is the angle between the radii drawn to the extremities of any side; as AOB.

The *radius* is the radius of the circumscribed circle, OA.

The *apothem* is the radius of the inscribed circle, OF.

342. Cor. From the equal $\triangle$s OAB, OBC, etc., we have

$$\angle AOB = \angle BOC = \angle COD, \text{ etc.} \qquad (?)$$

But the sum of these $\angle$s is four rt. $\angle$s. (§ 35)

Whence, *the angle at the centre of a regular polygon is equal to four right angles divided by the number of sides.*

EXERCISES.

Find the angle, and the angle at the centre,

1. Of a regular pentagon.

2. Of a regular dodecagon.

3. Of a regular polygon of 32 sides.

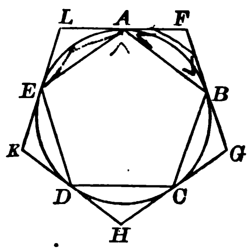
4. Of a regular polygon of 25 sides.

PROP. II. THEOREM.

343. *If the circumference of a circle be divided into any number of equal arcs,*

I. *Their chords form a regular inscribed polygon.*

II. *Tangents at the points of division form a regular circumscribed polygon.*



Given circumference ACD divided into five equal arcs, AB, BC, CD, etc., and chords AB, BC, etc.

Also, lines LF, FG, etc., tangent to $\odot\, ACD$ at A, B, etc., respectively, forming polygon $FGHKL$.

To Prove polygons $ABCDE$ and $FGHKL$ regular.

Proof. Chord $AB =$ chord $BC =$ chord CD, etc. (§ 158)

Again, arc $BCDE =$ arc $CDEA =$ arc $DEAB$, etc., for each is the sum of three of the equal arcs AB, BC, etc.

$\therefore\ \angle EAB = \angle ABC = \angle BCD$, etc. (§ 193)

Therefore, polygon $ABCDE$ is regular. (§ 339)

Again, in $\triangle$ ABF, BCG, CDH, etc., we have

$$AB = BC = CD, \text{ etc.}$$

Also, since arc $AB =$ arc $BC =$ arc CD, etc., we have

$\angle BAF = \angle ABF = \angle CBG = \angle BCG$, etc. (§ 197)

Whence, ABF, BCG, etc., are equal isosceles $\triangle$. (§§ 68, 96)

$$\therefore \angle F = \angle G = \angle H, \text{ etc.,}$$

and $$BF = BG = CG = CH, \text{ etc.} \qquad (\S\ 66)$$

$$\therefore FG = GH = HK, \text{ etc.}$$

Therefore, polygon $FGHKL$ is regular. (?)

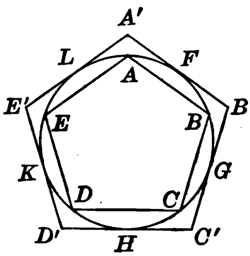
344. Cor. I. 1. *If from the middle point of each arc subtended by a side of a regular inscribed polygon lines be drawn to its extremities, a regular inscribed polygon of double the number of sides is formed.*

2. *If at the middle point of each arc included between two consecutive points of contact of a regular circumscribed polygon tangents be drawn, a regular circumscribed polygon of double the number of sides is formed.*

345. Cor. II. *An equilateral polygon inscribed in a circle is regular;* for its sides subtend equal arcs. (?)

PROP. III. THEOREM.

346. *Tangents to a circle at the middle points of the arcs subtended by the sides of a regular inscribed polygon, form a regular circumscribed polygon.*



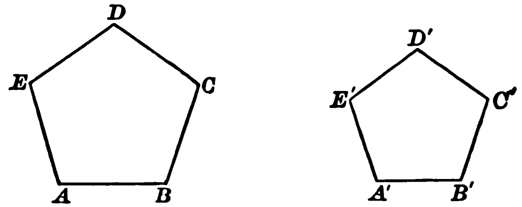
Given $ABCDE$ a regular polygon inscribed in $\odot AC$, and $A'B'C'D'E'$ a polygon whose sides $A'B'$, $B'C'$, etc., are tangent to $\odot AC$ at the middle points F, G, etc., of arcs AB, BC, etc., respectively.

To Prove $A'B'C'D'E'$ a regular polygon.

(**Arc** AF = arc BF = **arc** BG = arc CG, etc., and the proposition follows by § 343, II.)

PROP. IV. THEOREM.

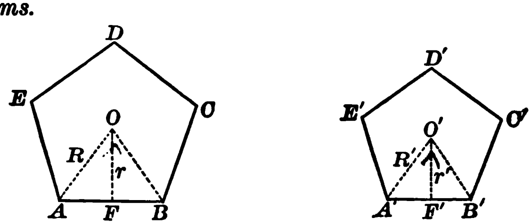
347. *Regular polygons of the same number of sides are similar.*



(The polygons fulfil the conditions of similarity given in § 252.)

PROP. V. THEOREM.

348. *The perimeters of two regular polygons of the same number of sides are to each other as their radii, or as their apothems.*



Given P and P' the perimeters, R and R' the radii, and r and r' the apothems, respectively, of regular polygons AC and $A'C'$ of the same number of sides.

To Prove $$\frac{P}{P'}=\frac{R}{R'}=\frac{r}{r'}.$$

Proof. Let O be the centre of polygon AC, and O' of $A'C'$, and draw lines OA, OB, $O'A'$, and $O'B'$.

Also, draw line $OF \perp AB$, and line $O'F' \perp A'B'$.

Then, $OA = R$, $O'A' = R'$, $OF = r$, and $O'F' = r'$.

Now in isosceles $\triangle$ OAB and $O'A'B'$,

$$\angle AOB = \angle A'O'B'. \qquad \textbf{(§ 342)}$$

And since $OA = OB$ and $O'A' = O'B'$, we have

$$\frac{OA}{O'A'} = \frac{OB}{O'B'}.$$

Therefore, $\triangle$ OAB and $O'A'B'$ are similar. (§ 261)

$$\therefore \frac{AB}{A'B'} = \frac{R}{R'} = \frac{r}{r'}. \quad \text{(§§ 253, II, 264)}$$

But polygons AC and $A'C'$ are similar. (§ 347)

$$\therefore \frac{P}{P'} = \frac{AB}{A'B'}. \quad \text{(§ 268)}$$

$$\therefore \frac{P}{P'} = \frac{R}{R'} = \frac{r}{r'}. \quad \text{(?)}$$

349. Cor. Let K denote the area of polygon AC, and K' of $A'C'$.

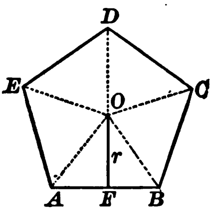
$$\therefore \frac{K}{K'} = \frac{\overline{AB}^2}{\overline{A'B'}^2}. \quad \text{(§ 322)}$$

But, $\frac{AB}{A'B'} = \frac{R}{R'} = \frac{r}{r'}$; whence, $\frac{K}{K'} = \frac{R^2}{R'^2} = \frac{r^2}{r'^2}$.

That is, *the areas of two regular polygons of the same number of sides are to each other as the squares of their radii, or as the squares of their apothems.*

PROP. VI. THEOREM.

350. *The area of a regular polygon is equal to one-half the product of its perimeter and apothem.*



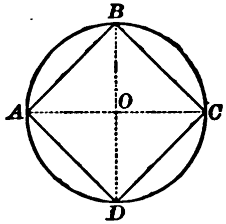
Given the perimeter equal to P, and the apothem OF equal to r, of regular polygon AC.

To Prove area $AC = \frac{1}{2} P \times r$.

($\triangle$ OAB, OBC, etc., have the common altitude r.)

PROP. VII. PROBLEM.

351. *To inscribe a square in a given circle.*



Given $\odot$ AC.

Required to inscribe a square in $\odot$ AC.

Construction. Draw diameters AC and BD $\perp$ to each other, and chords AB, BC, CD, and DA.

Then, $ABCD$ is an inscribed square.

(The proof is left to the pupil; see § 343, I.)

352. Cor. Denoting radius OA by R, we have

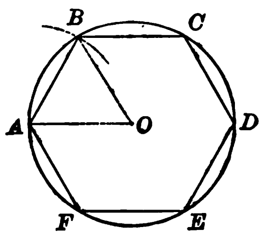
$$\overline{AB}^2 = \overline{OA}^2 + \overline{OB}^2 = 2R^2. \qquad (\S\ 272)$$

$$\therefore AB = R\sqrt{2}.$$

That is, *the side of an inscribed square is equal to the radius of the circle multiplied by* $\sqrt{2}$.

PROP. VIII. PROBLEM.

353. *To inscribe a regular hexagon in a given circle.*



Given $\odot$ AC.

Required to inscribe a regular hexagon in $\odot\, AC$.

Construction. Draw any radius OA.

With A as a centre, and AO as a radius, describe an arc cutting the given circumference at B, and draw chord AB.

Then, AB is a side of a regular inscribed hexagon.

Hence, to inscribe a regular hexagon in a given $\odot$, apply the radius six times as a chord.

Proof. Draw radius OB; then, $\triangle\, OAB$ is equilateral. (?)

Therefore, $\triangle\, OAB$ is equiangular. (§ 95)

Whence, $\angle\, AOB$ is one-third of two rt. $\angle$s. (?)

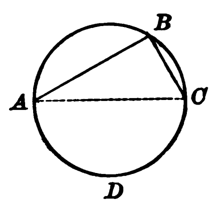
Then, $\angle\, AOB$ is one-sixth of four rt. $\angle$s, and arc AB is one-sixth of the circumference. (§ 154)

Then, AB is a side of a regular inscribed hexagon. (§ 343, I.)

354. Cor. I. *The side of a regular inscribed hexagon is equal to the radius of the circle.*

355. Cor. II. *If chords be drawn joining the alternate vertices of a regular inscribed hexagon, there is formed an inscribed equilateral triangle.*

356. Cor. III. *The side of an inscribed equilateral triangle is equal to the radius of the circle multiplied by* $\sqrt{3}$.



Given AB a side of an equilateral $\triangle$ inscribed in $\odot\, AD$ whose radius is R.

To Prove $AB = R\sqrt{3}$.

Proof. Draw diameter AC, and chord BC; then, BC is a side of a regular inscribed hexagon. (§ 355)

Now ABC is a rt. $\triangle$. (§ 195)

$$\therefore\ \overline{AB}^2 = \overline{AC}^2 - \overline{BC}^2 \quad (?)$$

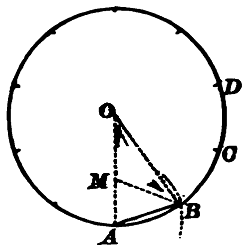
$$= (2\,R)^2 - R^2 \quad (\S\ 354)$$

$$= 4\,R^2 - R^2 = 3\,R^2.$$

$$\therefore\ AB = R\sqrt{3}.$$

PROP. IX. PROBLEM.

357. *To inscribe a regular decagon in a given circle.*



Given $\odot AC$.

Required to inscribe a regular decagon in $\odot AC$.

Construction. Draw any radius OA, and divide it internally in extreme and mean ratio at M (§ 297), so that

$$OA : OM = OM : AM. \qquad (1)$$

With A as a centre, and OM as a radius, describe an arc cutting the given circumference at B, and draw chord AB.

Then, AB is a side of a regular inscribed decagon.

Hence, to inscribe a regular decagon in a given $\odot$, divide the radius internally in extreme and mean ratio, and apply the greater segment ten times as a chord.

Proof. Draw lines OB and BM.

In $\triangle$s OAB and ABM, $\angle A = \angle A$.

And since, by cons., $OM = AB$, the proportion (1) becomes

$$OA : AB = AB : AM.$$

Therefore, $\triangle$s OAB and ABM are similar. (§ 261)

$$\therefore \angle ABM = \angle AOB. \qquad (?)$$

Again, $\triangle OAB$ is isosceles. (?)

Hence, the similar $\triangle ABM$ is isosceles, and

$$AB = BM = OM. \qquad (\text{Ax. } 1)$$

$$\therefore \angle OBM = \angle AOB. \qquad (?)$$

$$\therefore \angle ABM + \angle OBM = \angle AOB + \angle AOB.$$

Or, $\angle OBA = 2\angle AOB.$ (2)

But since $\triangle OAB$ is isosceles,

$2\angle OBA + \angle AOB = 180°.$ (§ 84)

Then, by (2), $5\angle AOB = 180°$, or $\angle AOB = 36°.$

Therefore, $\angle AOB$ is one-tenth of four rt. $\angle$s, and AB is a side of a regular inscribed decagon. (?)

358. Cor. I. *If chords be drawn joining the alternate vertices of a regular inscribed decagon, there is formed a regular inscribed pentagon.*

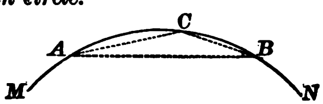
359. Cor. II. Denoting the radius of the $\odot$ by R, we have

$$AB = OM = \frac{R(\sqrt{5}-1)}{2}. \qquad (§ 298)$$

This is an expression for the side of a regular inscribed decagon in terms of the radius of the circle.

PROP. X. PROBLEM.

360. *To construct the side of a regular pentedecagon inscribed in a given circle.*



Given arc MN.

Required to construct the side of a regular inscribed polygon of fifteen sides.

Construction. Construct chord AB a side of a regular inscribed hexagon (§ 353), and chord AC a side of a regular inscribed decagon (§ 357), and draw chord BC.

Then, BC is a side of a regular inscribed pentedecagon.

Proof. By cons., arc BC is $\frac{1}{6} - \frac{1}{10}$, or $\frac{1}{15}$, of the circumference.

Hence, chord BC is a side of a regular inscribed pentedecagon. (?)

361. Sch. I. By bisecting arcs AB, BC, etc., in the figure of Prop. VII., we may construct a regular inscribed octagon (§ 343, I.); and by continuing the bisection, we may construct regular inscribed polygons of 16, 32, 64, etc., sides.

In like manner, by aid of Props. VIII., IX., and X., we may construct regular inscribed polygons of 12, 24, 48, etc., or of 20, 40, 80, etc., or of 30, 60, 120, etc., sides.

362. Sch. II. By drawing tangents to the circumference at the vertices of any one of the above inscribed regular polygons, we may construct a regular circumscribed polygon of the same number of sides. (§ 343, II.)

EXERCISES.

5. The angle at the centre of a regular polygon is the supplement of the angle of the polygon. (§ 127.)

6. The circumference of a circle is greater than the perimeter of any inscribed polygon.

7. An equiangular polygon circumscribed about a circle is regular. (§ 202.)

If r represents the radius, a the apothem, s the side, and k the area,

8. In an equilateral triangle, $a = \frac{1}{2} r$, and $k = \frac{3}{4} r^2\sqrt{3}$.

9. In a square, $a = \frac{1}{2} r\sqrt{2}$, and $k = 2 r^2$.

10. In a regular hexagon, $a = \frac{1}{2} r\sqrt{3}$, and $k = \frac{3}{2} r^2\sqrt{3}$.

11. In an equilateral triangle, $r = 2\,a$, $s = 2\,a\sqrt{3}$, and $k = 3\,a^2\sqrt{3}$.

12. In a square, $r = a\sqrt{2}$, $s = 2\,a$, and $k = 4\,a^2$.

13. In a regular hexagon, $r = \frac{2}{3}\,a\sqrt{3}$, and $k = 2\,a^2\sqrt{3}$.

14. In an equilateral triangle, express r, a, and k in terms of s.

15. In a square, express r, a, and k in terms of s.

16. In a regular hexagon, express a and k in terms of s.

17. In an equilateral triangle, express r, a, and s in terms of k.

18. In a square, express r, a, and s in terms of k.

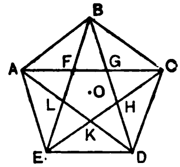
19. In a regular hexagon, express r and a in terms of k.

20. The apothem of an equilateral triangle is one-third the altitude of the triangle.

21. The sides of a regular polygon circumscribed about a circle are bisected at the points of contact. (§ 94.)

22. The radius drawn from the centre of a regular polygon to any vertex bisects the angle at that vertex. (§ 44.)

23. The diagonals of a regular pentagon are equal. (§ 63.)



24. The figure bounded by the five diagonals of a regular pentagon is a regular pentagon.

(Prove, by aid of § 164, that a ⊙ can be inscribed in $FGHKL$; then use Ex. 7, p. 198.)

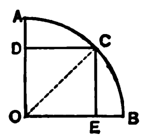
25. The area of a regular inscribed hexagon is a mean proportional between the areas of an inscribed, and of a circumscribed equilateral triangle.

(Prove, by aid of Exs. 8, 10, and 11, p. 198, that the product of the areas of the inscribed and circumscribed equilateral $\triangle$ is equal to the square of the area of the regular hexagon.)

26. If the diagonals AC and BE of regular pentagon $ABCDE$ intersect at F, prove $BE = AE + AF$. (Ex. 23.)

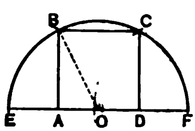
27. In the figure of Prop. IX., prove that OM is the side of a regular pentagon inscribed in a circle which is circumscribed about triangle OBM.

($\angle OBM = 36°$.)



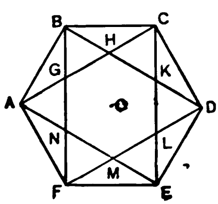
28. The area of the square inscribed in a sector whose central angle is a right angle is equal to one-half the square of the radius.

(To prove area $ODCE = \frac{1}{2}\overline{OC}^2$.)



29. The square inscribed in a semicircle is equivalent to two-fifths of the square inscribed in the entire circle.

(By Ex. 9, p. 198, the area of the square inscribed in the entire ⊙ is $2\,\overline{OB}^2$; we then have to prove area $ABCD = \frac{2}{5}$ of $2\,\overline{OB}^2 = \frac{4}{5}\,\overline{OB}^2$.)



30. The diagonals AC, BD, CE, DF, EA, and FB, of regular hexagon $ABCDEF$, form a regular hexagon whose area is equal to one-third the area of $ABCDEF$.

(The apothem of $GHKLMN$ is equal to the apothem of $\triangle ACE$, which may be found by Ex. 8, p. 198.)

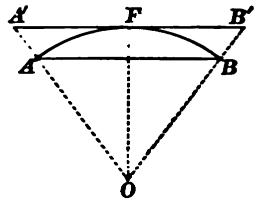
MEASUREMENT OF THE CIRCLE.

PROP. XI. THEOREM.

363. *If a regular polygon be inscribed in, or circumscribed about, a circle, and the number of its sides be indefinitely increased,*

I. *Its perimeter approaches the circumference as a limit.*

II. *Its area approaches the area of the circle as a limit.*



Given p and P the perimeters, and k and K the areas, of two regular polygons of the same number of sides respectively inscribed in, and circumscribed about, a ⊙.

Let C denote the circumference, and S the area, of the ⊙.

I. **To Prove** that, if the number of sides of the polygons be indefinitely increased, P and p approach the limit C.

Proof. Let $A'B'$ be a side of the polygon whose perimeter is P, and draw radius OF to its point of contact.

Also, draw lines OA' and OB' cutting the circumference at A and B, respectively, and chord AB.

Then, AB is a side of the polygon whose perimeter is p. (§ 342)

Now the two polygons are similar. (§ 347)

$$\therefore\ P:p = OA' : OF. \qquad (§\ 348)$$

$$\therefore\ P-p:p = OA' - OF : OF. \qquad (?)$$

$$\therefore\ (P-p) \times OF = p \times (OA' - OF). \qquad (?)$$

$$\therefore\ P-p = \frac{p}{OF} \times (OA' - OF).$$

But p is always $<$ the circumference of the $\odot$. (Ax. 4)

Also, $OA' - OF$ is $< A'F$. (§ 62)

$$\therefore\ P - p < \frac{C}{OF} \times A'F. \qquad (1)$$

Now, if the number of sides of each polygon be indefinitely increased, the polygons continuing to have the same number of sides, the length of each side will be indefinitely diminished, and $A'F$ will approach the limit 0.

Then, by (1), since $\frac{C}{OF}$ is a constant, $P - p$ will approach the limit 0.

But the circumference of the $\odot$ is $<$ the perimeter of the circumscribed polygon; * and it is $>$ the perimeter of the inscribed polygon. (Ax. 4)

Then the difference between each perimeter and the circumference, or $P - C$ and $C - p$, will approach the limit 0.

Therefore, P and p will each approach the limit C.

II. **To Prove** that K and k approach the limit S.

Proof. Since the given polygons are similar,

$$K : k = \overline{OA'}^2 : \overline{OF}^2. \qquad (\S\ 349)$$

$$\therefore\ K - k : k = \overline{OA'}^2 - \overline{OF}^2 : \overline{OF}^2. \qquad (?)$$

$$\therefore\ (K - k) \times \overline{OF}^2 = k \times (\overline{OA'}^2 - \overline{OF}^2). \qquad (?)$$

$$\therefore\ K - k = \frac{k}{\overline{OF}^2} \times (\overline{OA'}^2 - \overline{OF}^2) = \frac{k}{\overline{OF}^2} \times \overline{A'F}^2. \qquad (?)$$

Now, if the number of sides of each polygon be indefinitely increased, the polygons continuing to have the same number of sides, $A'F$ will approach the limit 0.

Then, $\frac{k}{\overline{OF}^2} \times \overline{A'F}^2$, being always $< \frac{S}{\overline{OF}^2} \times \overline{A'F}^2$, will approach the limit 0.

Whence, $K - k$ will approach the limit 0.

But the area of the $\odot$ is evidently $< K$, and $> k$.

Then, $K - S$ and $S - k$ will each approach the limit 0.

Therefore, K and k will each approach the limit S.

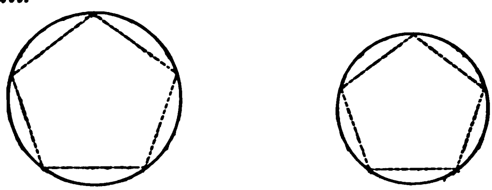
* For a rigorous proof of this statement, see Appendix, p. 386.

364. Cor. 1. *If a regular polygon be inscribed in a circle, and the number of its sides be indefinitely increased, its apothem approaches the radius of the circle as a limit.*

2. *If a regular polygon be circumscribed about a circle, and the number of its sides be indefinitely increased, its radius approaches the radius of the circle as a limit.*

Prop. XII. Theorem.

365. *The circumferences of two circles are to each other as their radii.*



Given C and C' the circumferences of two ⓢ whose radii are R and R', respectively.

To Prove $$\frac{C}{C'}=\frac{R}{R'}.$$

Proof. Inscribe in the ⓢ regular polygons of the same number of sides; P and P' being the perimeters of the polygons inscribed in ⓢ whose radii are R and R', respectively.

$$\therefore P : P' = R : R'. \qquad (\S\ 348)$$

$$\therefore P \times R' = P' \times R. \qquad (?)$$

Now let the number of sides of each inscribed polygon be indefinitely increased, the two polygons continuing to have the same number of sides.

Then, $P \times R'$ will approach the limit $C \times R'$,

and $P' \times R$ will approach the limit $C' \times R$. (§ 363, I.)

By the Theorem of Limits, these limits are equal. (?)

$$\therefore C \times R' = C' \times R, \text{ or } \frac{C}{C'}=\frac{R}{R'}. \qquad (\S\ 234)$$

366. Cor. I. Multiplying the terms of the ratio $\frac{R}{R'}$ by 2, we have

$$\frac{C}{C'}=\frac{2R}{2R'}.$$

Now let D and D' denote the diameters of the ⓢ whose radii are R and R', respectively.

$$\therefore \frac{C}{C'}=\frac{D}{D'}. \qquad (1)$$

That is, *the circumferences of two circles are to each other as their diameters.*

367. Cor. II. The proportion (1) of § 366 may be written

$$\frac{C}{D}=\frac{C'}{D'}. \qquad (\S\ 235)$$

That is, *the ratio of the circumference of a circle to its diameter has the same value for every circle.*

This constant value is denoted by the symbol π.

$$\therefore \frac{C}{D}=\pi. \qquad (1)$$

It is shown by methods of higher mathematics that the ratio π is incommensurable; hence, its numerical value can only be obtained approximately.

Its value to the nearest fourth decimal place is 3.1416.

368. Cor. III. Equation (1) of § 367 gives

$$C=\pi D.$$

That is, *the circumference of a circle is equal to its diameter multiplied by π.*

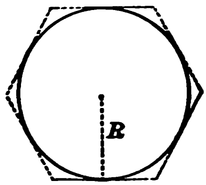
We also have $\qquad C=2\pi R.$

That is, *the circumference of a circle is equal to its radius multiplied by 2π.*

369. Def. In circles of different radii, *similar arcs, similar segments*, and *similar sectors* are those which correspond to equal central angles.

Prop. XIII. Theorem.

370. *The area of a circle is equal to one-half the product of its circumference and radius.*



Given R the radius, C the circumference, and S the area, of a ⊙.

To Prove $S = \frac{1}{2} C \times R$.

Proof. Circumscribe a regular polygon about the ⊙.

Let P denote its perimeter, and K its area.

Then since the apothem of the polygon is R,

$$K = \tfrac{1}{2} P \times R. \qquad (\S\ 350)$$

Now let the number of sides of the circumscribed polygon be indefinitely increased.

Then, K will approach the limit S,

and $\frac{1}{2} P \times R$ will approach the limit $\frac{1}{2} C \times R$. (§ 363)

By the Theorem of Limits, these limits are equal. (?)

$$\therefore\ S = \tfrac{1}{2} C \times R.$$

371. Cor. I. We have $C = 2\pi R$. (§ 368)

$$\therefore\ S = \pi R \times R = \pi R^2.$$

That is, *the area of a circle is equal to the square of its radius multiplied by* π.

Again, $S = \frac{1}{4}\pi \times 4R^2 = \frac{1}{4}\pi \times (2R)^2$.

Now let D denote the diameter of the ⊙.

$$\therefore\ S = \tfrac{1}{4}\pi D^2.$$

That is, *the area of a circle is equal to the square of its diameter multiplied by* $\frac{1}{4}\pi$.

372. Cor. II. Let S and S' denote the areas of two ⓢ whose radii are R and R', and diameters D and D', respectively.

$$\therefore \frac{S}{S'} = \frac{\pi R^2}{\pi R'^2} = \frac{R^2}{R'^2},$$

and $$\frac{S}{S'} = \frac{\frac{1}{4}\pi D^2}{\frac{1}{4}\pi D'^2} = \frac{D^2}{D'^2}. \qquad (\S\ 371)$$

That is, *the areas of two circles are to each other as the squares of their radii, or as the squares of their diameters.*

373. Cor. III. *The area of a sector is equal to one-half the product of its arc and radius.*

Given s and c the area and arc, respectively, of a sector of a ⊙ whose area, circumference, and radius are S, C, and R, respectively.

To Prove $$s = \tfrac{1}{2} c \times R.$$

Proof. A sector is the same part of the ⊙ that its arc is of the circumference.

$$\therefore \frac{s}{S} = \frac{c}{C}, \text{ or } s = c \times \frac{S}{C}.$$?

But, $$\frac{S}{C} = \tfrac{1}{2} R. \qquad (\S\ 370)$$

$$\therefore s = \tfrac{1}{2} c \times R.$$

374. Cor. IV. Since similar sectors are like parts of the ⓢ to which they belong (§ 369), it follows that

Similar sectors are to each other as the squares of their radii.

EXERCISES.

31. Find the circumference and area of a circle whose diameter is 5.

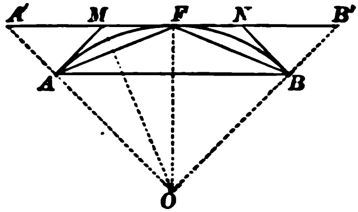
32. Find the radius and area of a circle whose circumference is 25π.

33. Find the diameter and circumference of a circle whose area is 289π.

34. The diameters of two circles are 64 and 88, respectively. What is the ratio of their areas?

Prop. XIV. Problem.

375. *Given p and P, the perimeters of a regular inscribed and of a regular circumscribed polygon of the same number of sides, to find p' and P', the perimeters of a regular inscribed and of a regular circumscribed polygon having double the number of sides.*



Solution. Let AB be a side of the polygon whose perimeter is p, and draw radius OF to middle point of arc AB.

Also, draw radii OA and OB cutting the tangent to the $\odot$ at F at points A' and B', respectively; then, $A'B'$ is a side of the polygon whose perimeter is P. (§ 342)

Draw chords AF and BF; also, draw AM and BN tangents to the $\odot$ at A and B, meeting $A'B'$ at M and N, respectively.

Then AF and MN are sides of the polygons whose perimeters are p' and P', respectively. (§ 344)

Hence, if n denotes the number of sides of the polygons whose perimeters are p and P, and therefore $2n$ the number of sides of the polygons whose perimeters are p' and P', we have

$$AB = \frac{p}{n},\ A'B' = \frac{P}{n},\ AF = \frac{p'}{2n},\ \text{and } MN = \frac{P'}{2n}. \qquad (1)$$

Draw line OM; then OM bisects $\angle A'OF$. (§ 175)

$$\therefore\ A'M : MF = OA' : OF. \qquad (§\ 249)$$

But OA' and OF are the radii of the polygons whose perimeters are P and p, respectively.

$$\therefore\ P : p = OA' : OF. \qquad (§\ 348)$$

$$\therefore\ P : p = A'M : MF. \qquad (?)$$

$$\therefore\ P + p : p = A'M + MF : MF. \qquad (?)$$

Or,
$$\frac{P+p}{p} = \frac{A'F}{MF} = \frac{\frac{1}{2}A'B'}{\frac{1}{2}MN}.$$

Then by (1),
$$\frac{P+p}{p} = \frac{\frac{P}{2n}}{\frac{P'}{4n}} = \frac{P}{2n} \times \frac{4n}{P'} = \frac{2P}{P'}.$$

Clearing of fractions,

$$P'(P+p) = 2P \times p.$$

$$\therefore\ P' = \frac{2P \times p}{P+p}. \qquad (2)$$

Again, in isosceles $\triangle$ ABF and AFM,

$$\angle ABF = \angle AFM. \qquad (\S\S\ 193, 197)$$

Therefore, $\triangle$ ABF and AFM are similar. (§ 255)

$$\therefore\ \frac{AF}{AB} = \frac{MF}{AF}. \qquad (?)$$

$$\therefore\ \overline{AF}^2 = AB \times MF. \qquad (?)$$

Then by (1),
$$\frac{p'^2}{4n^2} = \frac{p}{n} \times \frac{P'}{4n} = \frac{p \times P'}{4n^2}.$$

$$\therefore\ p'^2 = p \times P'.$$

$$\therefore\ p' = \sqrt{p \times P'}. \qquad (3)$$

Prop. XV. Problem.

376. *To compute an approximate value of* π (§ 367).

Solution. If the diameter of a $\odot$ is 1, the side of an inscribed square is $\frac{1}{2}\sqrt{2}$ (§ 352); hence, its perimeter is $2\sqrt{2}$.

Again, the side of a circumscribed square is equal to the diameter of the $\odot$; hence, its perimeter is 4.

We then put in equation (2), Prop. XIV.,

$$P = 4, \text{ and } p = 2\sqrt{2} = 2.82843.$$

$$\therefore\ P' = \frac{2\,P \times p}{P+p} = 3.31371.$$

We then put in equation (3), Prop. XIV.,

$$p = 2.82843, \text{ and } P' = 3.31371.$$

$$\therefore\ p' = \sqrt{p \times P'} = 3.06147.$$

These are the perimeters of the regular circumscribed and inscribed octagons, respectively.

Repeating the operation with these values, we put in (2),

$$P = 3.31371, \text{ and } p = 3.06147.$$

$$\therefore\ P' = \frac{2\,P \times p}{P+p} = 3.18260.$$

We then put in (3), $p = 3.06147$ and $P' = 3.18260$.

$$\therefore\ p' = \sqrt{p \times P'} = 3.12145.$$

These are, respectively, the perimeters of the regular circumscribed and inscribed polygons of sixteen sides.

In this way, we form the following table:

No. of Sides.	Perimeter of Reg. Circ. Polygon.	Perimeter of Reg. Insc. Polygon.
4	4.	2.82843
8	3.31371	3.06147
16	3.18260	3.12145
32	3.15172	3.13655
64	3.14412	3.14033
128	3.14222	3.14128
256	3.14175	3.14151
512	3.14163	3.14157

The last result shows that the circumference of a ⊙ whose diameter is 1 is $>$ 3.14157, and $<$ 3.14163.

Hence, an approximate value of π is 3.1416, correct to the fourth decimal place.

Note. The value of π to fourteen decimal places is

3.14159265358979. to 30

EXERCISES.

35. The area of a circle is equal to four times the area of the circle described upon its radius as a diameter.

36. The area of one circle is $2\frac{7}{9}$ times the area of another. If the radius of the first is 15, what is the radius of the second?

37. The radii of three circles are 3, 4, and 12, respectively. What is the radius of a circle equivalent to their sum?

38. Find the radius of a circle whose area is one-half the area of a circle whose radius is 9.

39. If the diameter of a circle is 48, what is the length of an arc of 85°?

40. If the radius of a circle is $3\sqrt{3}$, what is the area of a sector whose central angle is 152°?

41. If the radius of a circle is 4, what is the area of a segment whose arc is 120°? ($\pi = 3.1416$.)

(Subtract from the area of the sector whose central $\angle$ is 120°, the area of the isosceles $\triangle$ whose sides are radii and whose base is the chord of the segment.)

42. Find the area of the circle inscribed in a square whose area is 13.

43. Find the area of the square inscribed in a circle whose area is 196π.

44. If the apothem of a regular hexagon is 6, what is the area of its circumscribed circle?

45. If the length of a quadrant is 1, what is the diameter of the circle? ($\pi = 3.1416$.)

46. The length of the arc subtended by a side of a regular inscribed dodecagon is $\frac{4}{3}\pi$. What is the area of the circle?

47. The perimeter of a regular hexagon circumscribed about a circle is $12\sqrt{3}$. What is the circumference of the circle?

48. The area of a regular hexagon inscribed in a circle is $24\sqrt{3}$. What is the area of the circle?

49. The side of an equilateral triangle is 6. Find the areas of its inscribed and circumscribed circles.

50. The side of a square is 8. Find the circumferences of its inscribed and circumscribed circles.

51. Find the area of a segment having for its chord a side of a regular inscribed hexagon, if the radius of the circle is 10. ($\pi = 3.1416$.)

52. A circular grass-plot, 100 ft. in diameter, is surrounded by a walk 4 ft. wide. Find the area of the walk.

53. Two plots of ground, one a square and the other a circle, contain each 70686 sq. ft. How much longer is the perimeter of the square than the circumference of the circle? ($\pi = 3.1416$.)

54. A wheel revolves 55 times in travelling $\frac{1045\pi}{4}$ ft. What is its diameter in inches?

If r represents the radius, a the apothem, s the side, and k the area, prove that

55. In a regular octagon,

$$s = r\sqrt{2-\sqrt{2}},\ a = \tfrac{1}{2}r\sqrt{2+\sqrt{2}},\ \text{and}\ k = 2r^2\sqrt{2}. \quad (\S\ 375)$$

56. In a regular dodecagon,

$$s = r\sqrt{2-\sqrt{3}},\ a = \tfrac{1}{2}r\sqrt{2+\sqrt{3}},\ \text{and}\ k = 3r^2.$$

57. In a regular octagon,

$$s = 2a(\sqrt{2}-1),\ r = a\sqrt{4-2\sqrt{2}},\ \text{and}\ k = 8a^2(\sqrt{2}-1).$$

58. In a regular dodecagon,

$$s = 2a(2-\sqrt{3}),\ r = 2a\sqrt{2-\sqrt{3}},\ \text{and}\ k = 12a^2(2-\sqrt{3}).$$

59. In a regular decagon, $a = \frac{1}{4}r\sqrt{10+2\sqrt{5}}$. (§ 359.)
(Find the apothem by § 273.)

60. What is the number of degrees in an arc whose length is equal to that of the radius of the circle? ($\pi = 3.1416$.)
(Represent the number of degrees by x.)

61. Find the side of a square equivalent to a circle whose diameter is 3. ($\pi = 3.1416$.)

62. Find the radius of a circle equivalent to a square whose side is 10. ($\pi = 3.1416$.)

63. Given one side of a regular hexagon, to construct the hexagon.

64. Given one side of a regular pentagon, to construct the pentagon.
(Draw a ⊙ of any convenient radius, and construct a side of a regular inscribed pentagon.)

65. In a given square, to inscribe a regular octagon.
(Divide the angular magnitude about the centre of the square into eight equal parts.)

66. In a given equilateral triangle to inscribe a regular hexagon.

67. In a given sector whose central angle is a right angle, to inscribe a square.

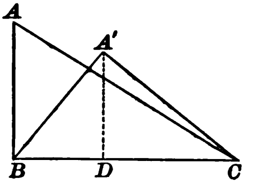
Note. For additional exercises on Book V., see p. 231.

APPENDIX TO PLANE GEOMETRY.

MAXIMA AND MINIMA OF PLANE FIGURES.

PROP. I. THEOREM.

377. *Of all triangles formed with two given sides, that in which these sides are perpendicular is the maximum.*



Given, in $\triangle ABC$ and $A'BC$, $AB = A'B$, and $AB \perp BC$.

To Prove area $ABC >$ area $A'BC$.

Proof. Draw $A'D \perp BC$; then,

$$A'B > A'D. \qquad (\S\ 46)$$

$$\therefore\ AB > A'D. \qquad (1)$$

Multiplying both members of (1) by $\frac{1}{2}BC$,

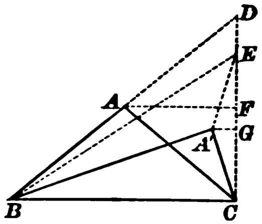
$$\tfrac{1}{2}BC \times AB > \tfrac{1}{2}BC \times A'D.$$

$$\therefore \text{ area } ABC > \text{area } A'BC. \qquad (\S\ 312)$$

378. Def. Two figures are said to be *isoperimetric* when they have equal perimeters.

PROP. II. THEOREM.

379. ***Of isoperimetric triangles having the same base, that which is isosceles is the maximum.***



Given ABC and $A'BC$ isoperimetric ▵, having the same base BC, and $\triangle ABC$ isosceles.

To Prove area $ABC >$ area $A'BC$.

Proof. Produce BA to D, making $AD = AB$, and draw line CD.

Then, $\angle BCD$ is a rt. $\angle$; for it can be inscribed in a semicircle, whose centre is A and radius AB. (§ 195)

Draw lines AF and $A'G \perp$ to CD; take point E on CD so that $A'E = A'C$, and draw line BE.

Then since ▵ ABC and $A'BC$ are isoperimetric,

$$AB + AC = A'B + A'C = A'B + A'E.$$

$$\therefore A'B + A'E = AB + AD = BD.$$

But, $A'B + A'E > BE$. (Ax. 4)

$$\therefore BD > BE.$$

$$\therefore CD > CE. \qquad (§ 51)$$

Now AF and $A'G$ are the ⊥s from the vertices to the bases of isosceles ▵ ACD and $A'CE$, respectively.

$$\therefore CF = \tfrac{1}{2} CD, \text{ and } CG = \tfrac{1}{2} CE. \qquad (§ 94)$$

$$\therefore CF > CG. \qquad (1)$$

Multiplying both members of (1) by $\frac{1}{2} BC$,

$$\tfrac{1}{2} BC \times CF > \tfrac{1}{2} BC \times CG.$$

$$\therefore \text{area } ABC > \text{area } A'BC. \qquad (?)$$

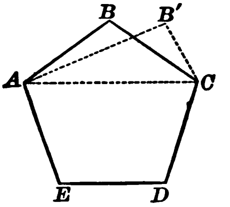
380. Cor. *Of isoperimetric triangles, that which is equilateral is the maximum.*

For if the maximum △ is not isosceles when any side is taken as the base, its area can be increased by making it isosceles. (§ 379)

Then, the maximum △ is equilateral.

Prop. III. Theorem.

381. *Of isoperimetric polygons having the same number of sides, that which is equilateral is the maximum.*



Given $ABCDE$ the maximum of polygons having the given perimeter and the given number of sides.

To Prove $ABCDE$ equilateral.

Proof. If possible, let sides AB and BC be unequal.

Let $AB'C$ be an isosceles △ with the base AC, having its perimeter equal to that of △ ABC.

$\therefore$ area $AB'C >$ area ABC. (§ 379)

Adding area $ACDE$ to both members,

area $AB'CDE >$ area $ABCDE$.

But this is impossible; for, by hyp., $ABCDE$ is the maximum of polygons having the given perimeter.

Hence, AB and BC cannot be unequal.

In like manner we have

$$BC = CD = DE, \text{ etc.}$$

Then, $ABCDE$ is equilateral.

PROP. IV. THEOREM.

382. *Of isoperimetric equilateral polygons having the same number of sides, that which is equiangular is the maximum.*

Given AB, BC, and CD any three consecutive sides of the maximum of isoperimetric equilateral polygons having the same number of sides.

To Prove $\angle ABC = \angle BCD$.

Proof. There may be three cases:

1. $ABC + BCD = 180°$. (Fig. 1.)
2. $ABC + BCD > 180°$. (Fig. 2.)
3. $ABC + BCD < 180°$. (Fig. 3.)

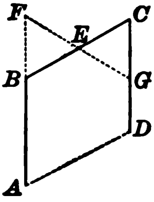


Fig. 1.

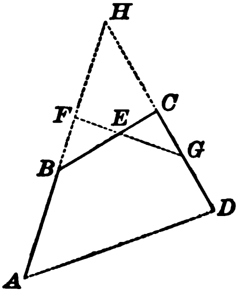


Fig. 2.

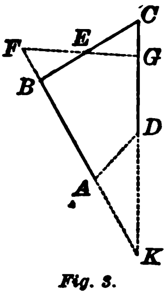


Fig. 3.

If possible, let $\angle ABC$ be $> \angle BCD$, and draw line AD.

In Fig. 1.

Let E be the middle point of BC; and draw line EF, meeting AB produced at F, making $EF = BE$.

Produce FE to meet CD at G.

Then in △ BEF and CEG, by hyp., $BE = CE$.

Also, $\angle BEF = \angle CEG$. (?)

And, $\angle EBF = \angle C$,

for each is the supplement of $\angle B$. (§ 33, 2)

$\therefore \triangle BEF = \triangle CEG$. (§§ 86, 68)

$\therefore BE = EF = CE = EG$, and $BF = CG$. (§ 66)

In Fig. 2.

Produce AB and DC to meet at H.

Since, by hyp., $\angle ABC > \angle BCD$, $\angle CBH < \angle BCH$.

$\therefore BH > CH$. (§ 99)

Lay off, on BH, $FH = CH$; and on DH, $GH = BH$; and draw line FG cutting BC at E.

$\therefore \triangle FGH = \triangle BCH$. (§ 63)

$\therefore \angle CBH = \angle FGH$. (§ 66)

Then, in $\triangle$ BEF and CEG, $\angle EBF = \angle CGE$.

Also, $\angle BEF = \angle CEG$. (?)

And $BF = CG$,

since $BF = BH - FH$, and $CG = GH - CH$.

$\therefore \triangle BEF = \triangle CEG$. (§§ 86, 68)

$\therefore BE = CE$ and $EF = EG$. (§ 66)

In Fig. 3.

Produce BA and CD to meet at K.

Since, by hyp., $\angle ABC > \angle BCD$, $CK > BK$. (?)

Lay off, on KB produced, $FK = CK$; and on CK, $GK = BK$; and draw line FG cutting BC at E.

$\therefore \triangle BCK = \triangle FGK$. (?)

$\therefore \angle F = \angle C$. (?)

Then, in $\triangle$ BEF and CEG, $\angle F = \angle C$.

Also, $\angle BEF = \angle CEG$. (?)

And $BF = CG$,

since $BF = FK - BK$, and $CG = CK - GK$.

$\therefore \triangle BEF = \triangle CEG$. (?)

$\therefore BE = CE$ and $EF = EG$. (?)

Then since, in either figure, $BC + CG = BF + FG$, and $\triangle BEF = \triangle CEG$, quadrilateral $AFGD$ is isoperimetric with, and $\approx$ to, quadrilateral $ABCD$.

Calling the remainder of the given polygon P, it follows that the polygon composed of $AFGD$ and P is isoperimetric with, and $\approx$ to, the polygon composed of $ABCD$ and P; that is, the *given* polygon.

Then the polygon composed of $AFGD$ and P must be the maximum of polygons having the given perimeter and the given number of sides.

Hence, the polygon composed of $AFGD$ and P is equilateral. (§ 381)

But this is impossible, since AF is $> DG$.

Hence, $\angle ABC$ cannot be $> \angle BCD$.

In like manner, $\angle ABC$ cannot be $< \angle BCD$.

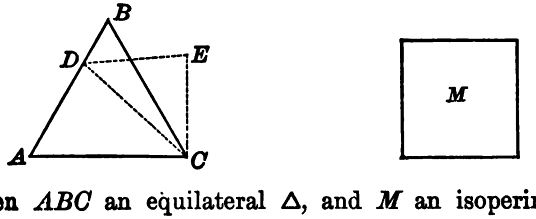
$$\therefore \angle ABC = \angle BCD.$$

Note. The case of *triangles* was considered in § 380. Fig. 3 also provides for the case of triangles by supposing D and K to coincide with A. In the case of *quadrilaterals*, $P = 0$.

383. Cor. *Of isoperimetric polygons having the same number of sides, that which is regular is the maximum.*

Prop. V. Theorem.

384. *Of two isoperimetric regular polygons, that which has the greater number of sides has the greater area.*



Given ABC an equilateral $\triangle$, and M an isoperimetric square.

To Prove area $M >$ area ABC.

Proof. Let D be any point in side AB of $\triangle ABC$.

Draw line DC; and construct isosceles $\triangle CDE$ isoperimetric with $\triangle BCD$, CD being its base.

$$\therefore \text{ area } CDE > \text{area } BCD. \qquad (\S\ 379)$$

$$\therefore \text{ area } ADEC > \text{area } ABC.$$

But, since $ADEC$ and M are isoperimetric,

$$\text{area } M > \text{area } ADEC. \qquad (\S\ 381)$$

$$\therefore \text{ area } M > \text{area } ABC.$$

In like manner, we may prove the area of a regular pentagon greater than that of an isoperimetric square; etc.

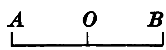
385. Cor. *The area of a circle is greater than the area of any polygon having an equal perimeter.*

SYMMETRICAL FIGURES.

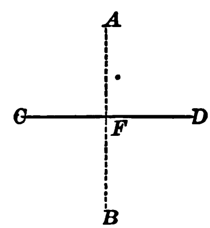
DEFINITIONS.

386. Two points are said to be *symmetrical* with respect to a third, called the *centre of symmetry*, when the latter bisects the straight line which joins them.

Thus, if O is the middle point of straight line AB, points A and B are symmetrical with respect to O as a centre.



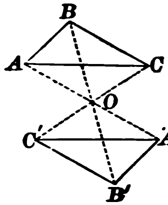
387. Two points are said to be *symmetrical* with respect to a straight line, called the *axis of symmetry*, when the latter bisects at right angles the straight line which joins them.



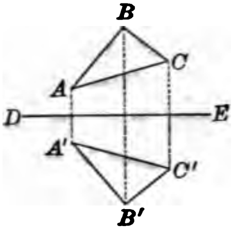
Thus, if line CD bisects line AB at right angles, points A and B are symmetrical with respect to CD as an axis.

388. Two figures are said to be symmetrical with respect to a centre, or with respect to an axis, when to every point of one there corresponds a symmetrical point in the other.

389. Thus, if to every point of triangle ABC there corresponds a symmetrical point of triangle $A'B'C'$, with respect to centre O, triangle $A'B'C'$ is symmetrical to triangle ABC with respect to centre O.



Again, if to every point of triangle ABC there corresponds a symmetrical point of triangle $A'B'C'$, with respect to axis DE, triangle $A'B'C'$ is symmetrical to triangle ABC with respect to axis DE.

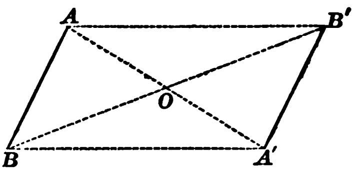


390. A figure is said to be symmetrical with respect to a centre when every straight line drawn through the centre cuts the figure in two points which are symmetrical with respect to that centre.

391. A figure is said to be symmetrical with respect to an axis when it divides it into two figures which are symmetrical with respect to that axis.

PROP. VI. THEOREM.

392. *Two straight lines which are symmetrical with respect to a centre are equal and parallel.*



Given str. lines AB and $A'B'$ symmetrical with respect to centre O.

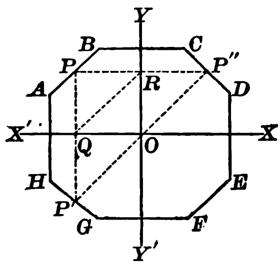
To Prove AB and $A'B'$ equal and $\parallel$.

Proof. Draw lines AA', BB', AB', and $A'B$.

Then, O bisects AA' and BB'. (§ 386)
Therefore, $AB'A'B$ is a $\square$. (§ 112)
Whence, AB and $A'B'$ are equal and $\parallel$. (?)

Prop. VII. Theorem.

393. *If a figure is symmetrical with respect to two axes at right angles to each other, it is symmetrical with respect to their intersection as a centre.*



Given figure AE symmetrical with respect to axes XX' and YY', intersecting each other at rt. $\angle$s at O.

To Prove AE symmetrical with respect to O as a centre.

Proof. Let P be any point in the perimeter of AE.

Draw line $PQ \perp XX'$, and line $PR \perp YY'$.

Produce PQ and PR to meet the perimeter of AE at P' and P'', respectively, and draw lines QR, OP', and OP''.

Then since AE is symmetrical with respect to XX',

$$PQ = P'Q. \qquad (\S\ 387)$$

But $PQ = OR$; whence, OR is equal and $\parallel$ to $P'Q$.

Therefore, $OP'QR$ is a $\square$. (?)

Whence, QR is equal and $\parallel$ to OP'. (?)

In like manner, we may prove $OP''RQ$ a $\square$; and therefore QR equal and $\parallel$ to OP''.

Then since both OP' and OP'' are equal and $\parallel$ to QR, $P'OP''$ is a str. line which is bisected at O.

That is, every str. line drawn through O is bisected at that point, and hence AE is symmetrical with respect to O as a centre. (§ 390)

ADDITIONAL EXERCISES.

BOOK I.

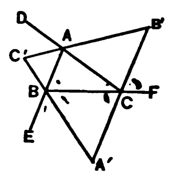
1. Every point within an angle, and not in the bisector, is unequally distant from the sides of the angle.
(Prove by *Reductio ad Absurdum*.)

2. If two lines are cut by a third, and the sum of the interior angles on the same side of the transversal is less than two right angles, the lines will meet if sufficiently produced.
(Prove by *Reductio ad Absurdum*.)

3. State and prove the converse of Prop. XXXVII., II.
(Prove $\angle BAD + \angle B = 180°$.)

4. The bisectors of the exterior angles of a triangle form a triangle whose angles are respectively the half-sums of the angles of the given triangle taken two and two. (Ex. 69, p. 67.)

(To prove $\angle A' = \frac{1}{2}(\angle ABC + \angle BCA)$, etc.)



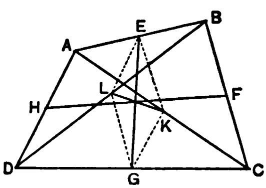
5. If CD is the perpendicular from C to side AB of triangle ABC, and CE the bisector of angle C, prove $\angle DCE$ equal to one-half the difference of angles A and B.

6. If E, F, G, and H are the middle points of sides AB, BC, CD, and DA, respectively, of quadrilateral $ABCD$, prove $EFGH$ a parallelogram whose perimeter is equal to the sum of the diagonals of the quadrilateral. (§ 130.)

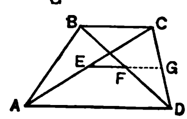
7. The lines joining the middle points of the opposite sides of a quadrilateral bisect each other. (Ex. 6, p. 220.)

8. The lines joining the middle points of the opposite sides of a quadrilateral bisect the line joining the middle points of the diagonals.

($EKGL$ is a $\square$, and its diagonals bisect each other.)



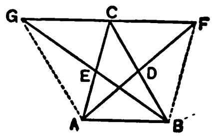
9. The line joining the middle points of the diagonals of a trapezoid is parallel to the bases and equal to one-half their difference.



10. If D is any point in side AC of triangle ABC, and E, F, G, and H the middle points of AD, CD, BC, and AB, respectively, prove $EFGH$ a parallelogram.

11. If E and G are the middle points of sides AB and CD, respectively, of quadrilateral $ABCD$, and K and L the middle points of diagonals AC and BD, respectively, prove $\triangle EKL = \triangle GKL$.

12. If D and E are the middle points of sides BC and AC, respectively, of triangle ABC, and AD be produced to F and BE to G, making $DF = AD$ and $EG = BE$, prove that line FG passes through C, and is bisected at that point.



13. If D is the middle point of side BC of triangle ABC, prove $AD < \frac{1}{2}(AB + AC)$.

(Produce AD to E, making $DE = AD$.)

14. The sum of the medians of a triangle is less than the perimeter, and greater than the semi-perimeter of the triangle.

(Ex. 13, p. 221, and Ex. 106, p. 71.)

15. If the bisectors of the interior angle at C and the exterior angle at B of triangle ABC meet at D, prove $\angle BDC = \frac{1}{2}\angle A$.

16. If AD and BD are the bisectors of the exterior angles at the extremities of the hypotenuse of right triangle ABC, and DE and DF are drawn perpendicular, respectively, to CA and CB produced, prove $CEDF$ a square.

(D is equally distant from AC and BC.)

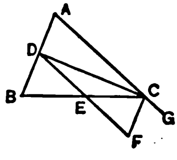
17. AD and BE are drawn from two of the vertices of triangle ABC to the opposite sides, making $\angle BAD = \angle ABE$; if $AD = BE$, prove the triangle isosceles.

18. If perpendiculars AE, BF, CG, and DH, be drawn from the vertices of parallelogram $ABCD$ to any line in its plane, not intersecting its surface, prove

$$AE + CG = BF + DH.$$

(The sum of the bases of a trapezoid is equal to twice the line joining the middle points of the non-parallel sides.)

19. If CD is the bisector of angle C of triangle ABC, and DF be drawn parallel to AC meeting BC at E and the bisector of the angle exterior to C at F, prove $DE = EF$.



20. If E and F are the middle points of sides AB and AC, respectively, of triangle ABC, and AD the perpendicular from A to BC, prove $\angle EDF = \angle EAF$. (Ex. 83, p. 69.)

21. If the median drawn from any vertex of a triangle is greater than, equal to, or less than one-half the opposite side, the angle at that vertex is acute, right, or obtuse, respectively. (§ 98.)

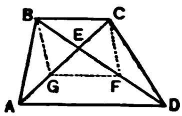
22. The number of diagonals of a polygon of n sides is $\frac{n(n-3)}{2}$.

23. The sum of the medians of a triangle is greater than three-fourths the perimeter of the triangle.

(Fig. of Prop. LII. Since $AO = \frac{2}{3} AD$ and $BO = \frac{2}{3} BE$, we have $AB < \frac{2}{3}(AD + BE)$, by Ax. 4.)

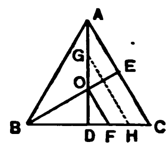
24. If the lower base AD of trapezoid $ABCD$ is double the upper base BC, and the diagonals intersect at E, prove $CE = \frac{1}{3} AC$ and $BE = \frac{1}{3} BD$.

(Let F be the middle point of DE, and G of AE.)



25. If O is the point of intersection of the medians AD and BE of equilateral triangle ABC, and line OF be drawn parallel to side AC, meeting side BC at F, prove that DF is equal to $\frac{1}{6} BC$. (§ 133.)

(Let G be the middle point of OA.)



26. If equiangular triangles be constructed on the sides of a triangle, the lines drawn from their outer vertices to the opposite vertices of the triangle are equal. (§ 63.)

27. If two of the medians of a triangle are equal, the triangle is isosceles.

(Fig. of Prop. LII. Let $AD = BE$.)

BOOK II.

28. AB and AC are the tangents to a circle from point A, and D is any point in the smaller of the arcs subtended by chord BC. If a tangent to the circle at D meets AB at E and AC at F, prove the perimeter of triangle AEF constant. (§ 174.)

29. The line joining the middle points of the arcs subtended by sides AB and AC of an inscribed triangle ABC cuts AB at F and AC at G. Prove $AF = AG$.

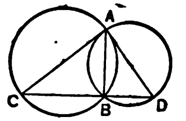
($\angle AFG = \angle AGF$.)

30. If $ABCD$ is a circumscribed quadrilateral, prove the angle between the lines joining the opposite points of contact equal to $\frac{1}{2}(A + C)$. (§ 202.)

31. If sides AB and BC of inscribed hexagon $ABCDEF$ are parallel to sides DE and EF, respectively, prove side AF parallel to side CD. (§ 172.)

(Draw line CF, and prove $\angle AFC = \angle FCD$.)

32. If AB is the common chord of two intersecting circles, and AC and AD diameters drawn from A, prove that line CD passes through B. (§ 195.)



33. If AB is a common exterior tangent to two circles which touch each other externally at C, prove $\angle ACB$ a right angle.

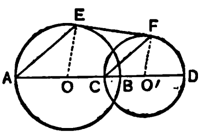
(Draw the common tangent at C, meeting AB at D.)

34. If AB and AC are the tangents to a circle from point A, and D is any point on the greater of the arcs subtended by chord BC, prove the sum of angles ABD and ACD constant.

35. If A, C, B, and D are four points in a straight line, B being between C and D, and EF is a common tangent to the circles described upon AB and CD as diameters, prove

$$\angle BAE = \angle DCF.$$

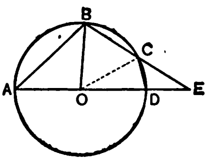
(We have $OE \parallel O'F$.)



36. $ABCD$ is an inscribed quadrilateral, AD being a diameter of the circle. If O is the centre, and sides AD and BC produced meet at E making $CE = OA$, prove

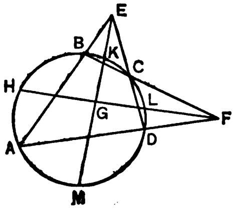
$$\angle AOB = 3\angle CED.$$

($\angle AOB$ is an ext. $\angle$ of $\triangle OBE$, and $\angle BCO$ of $\triangle OCE$.)



37. $ABCD$ is a quadrilateral inscribed in a circle. If sides AB and DC produced intersect at E, and sides AD and BC produced at F, prove the bisectors of angles E and F perpendicular. (§ 199.)

(Prove arc HM + arc $KL = 180°$.)



38. If $ABCD$ is an inscribed quadrilateral, and sides AD and BC produced meet at P, the tangent at P to the circle circumscribed about triangle ABP is parallel to CD. (§ 196.)

(Prove $\angle$ between the tangent and BP equal to $\angle PCD$.)

39. $ABCD$ is a quadrilateral inscribed in a circle. Another circle is described upon AD as a chord, meeting AB and CD at E and F, respectively. Prove chords BC and EF parallel.

(Prove $\angle ABC = \angle AEF$.)

40. If $ABCDEFGH$ is an inscribed octagon, the sum of angles A, C, E, and G is equal to six right angles. (§ 193.)

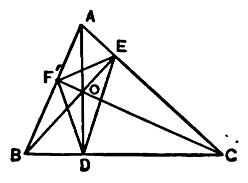
41. If the number of sides of an inscribed polygon is even, the sum of the alternate angles is equal to as many right angles as the polygon has sides less two.

(Use same method of proof as in Ex. 40.)

42. If a right triangle has for its hypotenuse the side of a square, and lies without the square, the straight line drawn from the centre of the square to the vertex of the right angle bisects the right angle. (§ 200.)

43. The perpendiculars from the vertices of a triangle to the opposite sides are the bisectors of the angles of the triangle formed by joining the feet of the perpendiculars.

(To prove AD, BE, and CF the bisectors of the $\angle$s of $\triangle DEF$. By § 200, a $\odot$ can be circumscribed about quadrilateral $BDOF$; then $\angle ODF = \angle OBF$; in this way, $\angle ODF = 90° - \angle BAC$.)

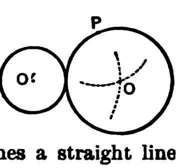


CONSTRUCTIONS.

44. Given a side, an adjacent angle, and the radius of the circumscribed circle of a triangle, to construct the triangle.

What restriction is there on the values of the given lines?

45. To describe a circle of given radius tangent to a given circle, and passing through a given point without the circle.



46. To draw between two given intersecting lines a straight line which shall be equal to one given straight line, and parallel to another.

(Draw a $\parallel$ to one of the intersecting lines.)

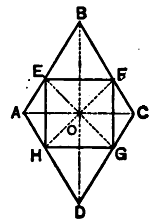
47. Given an angle of a triangle, the length of its bisector, and the length of the perpendicular from its vertex to the opposite side, to construct the triangle.

(The side opposite the given $\angle$ is tangent to a $\odot$ drawn with the vertex as a centre, and with the $\perp$ from the vertex to the opposite side as a radius.)

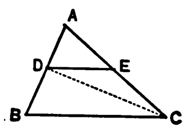
48. Given an angle of a triangle, and the segments of the opposite side made by the perpendicular from its vertex, to construct the triangle. (§ 226.)

49. To inscribe a square in a given rhombus.

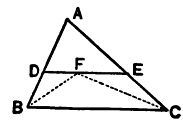
(Bisect the $\angle$s between diagonals AC and BD. To prove $EFGH$ a square, prove $\triangle$s OBE, OBF, ODG, and ODH equal; whence, $OE = OF = OG = OH$.)



50. To draw a parallel to side BC of triangle ABC meeting AB and AC in D and E, respectively, so that DE may equal EC.



51. To draw a parallel to side BC of triangle ABC, meeting AB and AC in D and E, respectively, so that DE may equal the sum of BD and CE.



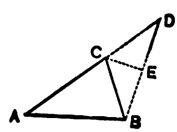
52. Given an angle of a triangle, the length of the perpendicular from the vertex of another angle to the opposite side, and the radius of the circumscribed circle, to construct the triangle.

(The centre of the circumscribed $\odot$ is equally distant from the given vertices.)

53. Through a given point without a given circle to draw a secant whose internal and external segments shall be equal. (Ex. 65, p. 103.)

54. Given the base of a triangle, an adjacent angle, and the sum of the other two sides, to construct the triangle.

(Lay off AD equal to the sum of the other two sides.)



38. If $ABCD$ is an inscribed quadrilateral, and sides AD and BC produced meet at P, the tangent at P to the circle circumscribed about triangle ABP is parallel to CD. (§ 196.)

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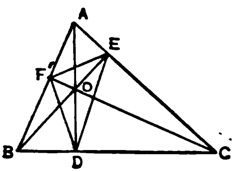
(Use same method of proof as in Ex. 40.)

42. If a right triangle has for its hypotenuse the side of a square, and lies without the square, the straight line drawn from the centre of the square to the vertex of the right angle bisects the right angle. (§ 200.)

43. The perpendiculars from the vertices of a triangle to the opposite sides are the bisectors of the angles of the triangle formed by joining the feet of the perpendiculars.



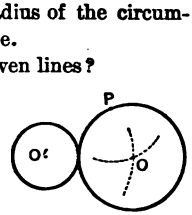
(To prove AD, BE, and CF the bisectors of the $\angle$s of $\triangle DEF$. By § 200, a $\odot$ can be circumscribed about quadrilateral $BDOF$; then $\angle ODF = \angle OBF$; in this way, $\angle ODF = 90° - \angle BAC$.)

Constructions.

44. Given a side, an adjacent angle, and the radius of the circumscribed circle of a triangle, to construct the triangle.

What restriction is there on the values of the given lines?

45. To describe a circle of given radius tangent to a given circle, and passing through a given point without the circle.



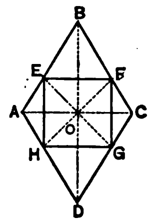
46. To draw between two given intersecting lines a straight line which shall be equal to one given straight line, and parallel to another.

(Draw a $\parallel$ to one of the intersecting lines.)

47. Given an angle of a triangle, the length of its bisector, and the length of the perpendicular from its vertex to the opposite side, to construct the triangle.

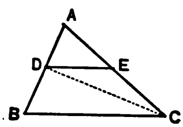
(The side opposite the given $\angle$ is tangent to a $\odot$ drawn with the vertex as a centre, and with the $\perp$ from the vertex to the opposite side as a radius.)

48. Given an angle of a triangle, and the segments of the opposite side made by the perpendicular from its vertex, to construct the triangle. (§ 226.)

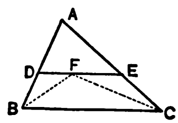


49. To inscribe a square in a given rhombus.

(Bisect the $\angle$s between diagonals AC and BD. To prove $EFGH$ a square, prove $\triangle$s OBE, OBF, ODG, and ODH equal; whence, $OE = OF = OG = OH$.)



50. To draw a parallel to side BC of triangle ABC meeting AB and AC in D and E, respectively, so that DE may equal EC.

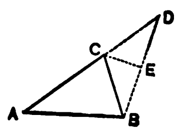


51. To draw a parallel to side BC of triangle ABC, meeting AB and AC in D and E, respectively, so that DE may equal the sum of BD and CE.

52. Given an angle of a triangle, the length of the perpendicular from the vertex of another angle to the opposite side, and the radius of the circumscribed circle, to construct the triangle.

(The centre of the circumscribed $\odot$ is equally distant from the given vertices.)

53. Through a given point without a given circle to draw a secant whose internal and external segments shall be equal. (Ex. 65, p. 103.)



54. Given the base of a triangle, an adjacent angle, and the sum of the other two sides, to construct the triangle.

(Lay off AD equal to the sum of the other two sides.)

38. If $ABCD$ is an inscribed quadrilateral, and sides AD and BC produced meet at P, the tangent at P to the circle circumscribed about triangle ABP is parallel to CD. (§ 196.)

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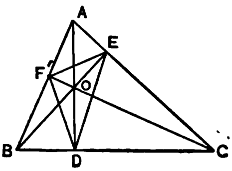

(To prove AD, BE, and CF the bisectors of the $\angle$s of $\triangle DEF$. By § 200, a $\odot$ can be circumscribed about quadrilateral $BDOF$; then $\angle ODF = \angle OBF$; in this way, $\angle ODF = 90° - \angle BAC$.)

Constructions.

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What restriction is there on the values of the given lines?

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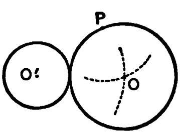

46. To draw between two given intersecting lines a straight line which shall be equal to one given straight line, and parallel to another.

(Draw a ∥ to one of the intersecting lines.)

47. Given an angle of a triangle, the length of its bisector, and the length of the perpendicular from its vertex to the opposite side, to construct the triangle.

(The side opposite the given $\angle$ is tangent to a $\odot$ drawn with the vertex as a centre, and with the $\perp$ from the vertex to the opposite side as a radius.)

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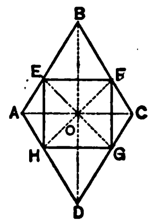
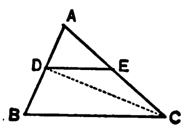
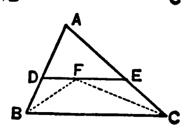


49. To inscribe a square in a given rhombus.

(Bisect the $\angle$s between diagonals AC and BD. To prove $EFGH$ a square, prove $\triangle$s OBE, OBF, ODG, and ODH equal; whence, $OE = OF = OG = OH$.)



50. To draw a parallel to side BC of triangle ABC meeting AB and AC in D and E, respectively, so that DE may equal EC.

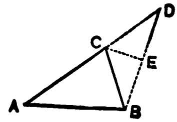


51. To draw a parallel to side BC of triangle ABC, meeting AB and AC in D and E, respectively, so that DE may equal the sum of BD and CE.

52. Given an angle of a triangle, the length of the perpendicular from the vertex of another angle to the opposite side, and the radius of the circumscribed circle, to construct the triangle.

(The centre of the circumscribed $\odot$ is equally distant from the given vertices.)

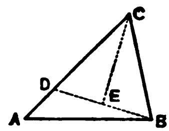
53. Through a given point without a given circle to draw a secant whose internal and external segments shall be equal. (Ex. 65, p. 103.)



54. Given the base of a triangle, an adjacent angle, and the sum of the other two sides, to construct the triangle.

(Lay off AD equal to the sum of the other two sides.)

55. Given the base of a triangle, an adjacent acute angle, and the difference of the other two sides, to construct the triangle.

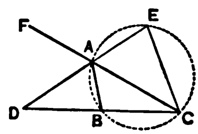


What restriction is there on the values of the given lines?

56. Given the feet of the perpendiculars from the vertices of a triangle to the opposite sides, to construct the triangle. (Ex. 43.)

BOOK III.

57. In any triangle, the product of any two sides is equal to the product of the segments of the third side formed by the bisector of the exterior angle at the opposite vertex, minus the square of the bisector.



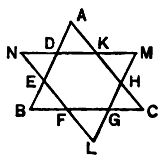
(To prove $AB \times AC = DB \times DC - \overline{AD}^2$. The work is carried out as in § 288; first prove $\triangle$ ABD and ACE similar.)

58. If the sides of a triangle are $AB = 4$, $AC = 5$, and $BC = 6$, find the length of the bisector of the exterior angle at vertex A. (§ 251.)

59. ABC is an isosceles triangle. If the perpendicular to AB at A meets base BC, produced if necessary, at E, and D is the middle point of BE, prove AB a mean proportional between BC and BD. (Ex. 83, p. 69.)

($\triangle$ ABC and ABD are similar.)

60. If D and E, F and G, and H and K are points on sides AB, BC, and CA, respectively, of triangle ABC, so taken that $AD = DE = EB$, $BF = FG = GC$, and $CH = HK = KA$, prove that lines EF, GH, and KD, when produced, form a triangle equal to ABC.



(By § 248, sides of $\triangle LMN$ are $\parallel$, respectively, to sides of $\triangle ABC$.)

61. The square of the common tangent to two circles which are tangent to each other externally is equal to 4 times the product of their radii. (§ 273.)

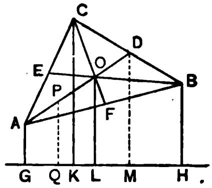
62. The sides AB and BC of triangle ABC are 3 and 7, respectively, and the length of the bisector of the exterior angle B is $3\sqrt{7}$. Find side AC. (Ex. 57, and § 251.)

63. One segment of a chord drawn through a point 7 units from the centre of a circle is 4 units. If the diameter of the circle is 15 units, what is the other segment? (§ 280.)

64. If E is the middle point of one of the parallel sides BC of trapezoid $ABCD$, and AE and DE produced meet DC and AB produced at F and G, respectively, prove FG parallel to AD.

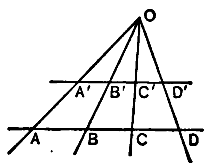
($\triangle$ ADG and BEG are similar, as also are $\triangle$ ADF and CEF.)

65. The perpendicular from the intersection of the medians of a triangle to any straight line in the plane of the triangle, not intersecting its surface, is equal to one-third the sum of the perpendiculars from the vertices of the triangle to the same line.



(The sum of the bases of a trapezoid is equal to twice the line joining the middle points of the non-parallel sides.)

66. If two parallels are cut by three or more straight lines passing through a common point, the corresponding segments are proportional.



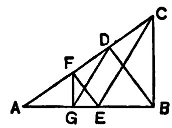
(To prove $\dfrac{AB}{A'B'} = \dfrac{BC}{B'C'} = \dfrac{CD}{C'D'}$. $\triangle$ OAB, OBC, and OCD are similar, respectively, to $\triangle$ $OA'B'$, $OB'C'$, and $OC'A'$.)

67. State and prove the converse of Ex. 66.

(Fig. of Ex. 66. To prove that AA', BB', CC', and DD' pass through a common point. Let AA' and BB' meet at O, and draw OC and OC'; then prove $\triangle$ OBC and $OB'C'$ similar.)

68. The non-parallel sides of a trapezoid and the line joining the middle points of the parallel sides, if produced, meet in a common point. (Ex. 67.)

69. BD is the perpendicular from the vertex of the right angle to the hypotenuse of right triangle ABC. If E is any point in AB, and EF be drawn perpendicular to AC, and FG perpendicular to AB, prove lines CE and DG parallel.



($\triangle$ ABC and AEF are similar. By § 271, 2, we may prove $AD : CD = \overline{AB}^2 : \overline{BC}^2$, and $AG : EG = \overline{AF}^2 : \overline{EF}^2$; then, we have $AD : CD = AG : EG$.)

70. In right triangle ABC, $\overline{BC}^2 = 3\,\overline{AC}^2$. If CD be drawn from the vertex of the right angle to the middle point of AB, prove $\angle ACD$ equal to 60°. (Ex. 83, p. 69.)

(Prove $AC = \frac{1}{2} AB$.)

71. If D is the middle point of side BC of right triangle ABC, and DE be drawn perpendicular to hypotenuse AB, prove

$$\overline{AE}^2 - \overline{BE}^2 = \overline{AC}^2.$$

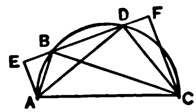
($AE = AB - BE$; square this by the rule of Algebra.)

72. If BE and CF are medians drawn from the extremities of the hypotenuse of right triangle ABC, prove

$$4\,\overline{BE}^2 + 4\,\overline{CF}^2 = 5\,\overline{BC}^2. \quad (\S\ 272.)$$

73. If ABC and ADC are angles inscribed in a semicircle, and AE and CF be drawn perpendicular to BD produced, prove

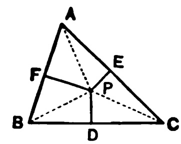
$$\overline{BE}^2 + \overline{BF}^2 = \overline{DE}^2 + \overline{DF}^2. \quad (\S\ 273.)$$



74. If perpendiculars PF, PD, and PE be drawn from any point P to sides AB, BC, and CA, respectively, of a triangle, prove

$$\overline{AF}^2 + \overline{BD}^2 + \overline{CE}^2 = \overline{AE}^2 + \overline{BF}^2 + \overline{CD}^2.$$

(§ 273.)



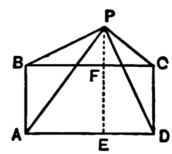
75. If BC is the hypotenuse of right triangle ABC, prove

$$(AB + BC + CA)^2 = 2(AB + BC)(BC + CA).$$

(Square $AB+BC+CA$ by the rule of Algebra.)

76. If lines be drawn from any point P to the vertices of rectangle $ABCD$, prove

$$\overline{PA}^2 + \overline{PC}^2 = \overline{PB}^2 + \overline{PD}^2.$$



77. If AB and AC are the equal sides of an isosceles triangle, and BD be drawn perpendicular to AC, prove $2\,AC \times CD = \overline{BC}^2$.

($AD = AC - CD$; square this by the rule of Algebra.)

78. If AD and BE are the perpendiculars from vertices A and B, respectively, of acute-angled triangle ABC to the opposite sides, prove

$$AC \times AE + BC \times BD = \overline{AB}^2.$$

(By § 277, $2\,AC \times AE = \overline{AB}^2 + \overline{AC}^2 - \overline{BC}^2$; and in like manner a value may be found for $2\,BC \times BD$.)

79. The sum of the squares of the sides of a parallelogram is equal to the sum of the squares of its diagonals. (§ 279, I.)

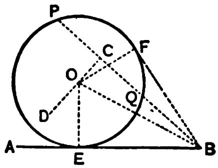
80. To construct a triangle similar to a given triangle, having given its perimeter.

(Divide the perimeter into parts proportional to the sides of the $\triangle$.)

81. To construct a right triangle, having given its perimeter and an acute angle.

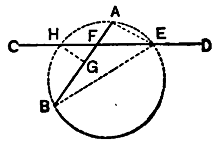
(From any point in one side of the given $\angle$ draw a $\perp$ to the other side.)

82. To describe a circle through two given points, tangent to a given straight line. (§ 282.)



(To prove $\odot$ drawn with O as a centre and OP as a radius tangent to AB, draw BF tangent to the $\odot$, and prove $\triangle\, OBE = \triangle\, OBF$.)

83. If A and B are points on either side of line CD, and line AB cuts CD at F, find a point E in CD such that



$$AE : BE = AF : BF. \quad (§\ 249.)$$

(EF bisects $\angle\, AEB$ of $\triangle\, ABE$.)

BOOK IV.

84. In the figure on p. 174,

(a) Prove lines CF and BH perpendicular.

(If CF and BH meet at S, $\angle\, CSH$ is an ext. $\angle$ of $\triangle\, BCS$.)

(b) Prove lines AG and BK parallel.

(c) Prove the sum of the perpendiculars from H and L to AB produced equal to AB.

(If $\perp$ from H meets BA produced at Q, $\triangle\, AHQ = \triangle\, ACD$.)

(d) Prove triangles AFH, BEL, and CGK each equivalent to ABC.

(If AF be taken as the base of $\triangle\, AFH$, its altitude is equal to CD.)

(e) Prove C, H, and L in the same straight line.

(Prove CH and CL in the same str. line.)

(f) Prove the square described upon the sum of AC and BC equivalent to the square described upon AB, plus 4 times $\triangle\, ABC$.

(Square $AC + BC$ by the rule of Algebra.)

(g) Prove the sum of angles AFH, AHF, BEL, and BLE equal to a right angle.

($\angle\, AFH + \angle\, AHF = 180° - \angle\, FAH$.)

(h) If FN and EP are the perpendiculars from F and E, respectively, to HA and LB produced, prove triangles AFN and BEP each equal to ABC.

(i) Prove $\overline{EL}^2 + \overline{FH}^2 + \overline{GK}^2 = 6\,\overline{AB}^2$.

(EL is the hypotenuse of rt. $\triangle\, ELP$, and FH of $\triangle\, FHN$; sides PL and HN may be found by (h).)

(*j*) Prove $\overline{CF}^2 - \overline{CE}^2 = \overline{AC}^2 - \overline{BC}^2$.

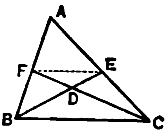
(*k*) Prove that lines AL, BH, and CM meet at a common point. (Ex. 84, (*a*).)

(Produce DC to T, making $CT = DM$, and prove AL, BH, and CM the ⊥s from the vertices to the opposite sides of $\triangle ABT$.)

(*l*) Prove that lines HG, LK, and MC when produced meet at a common point.

(Draw GT and KT, and prove ∆s CGT and CKT rt. ∆s.)

85. If BE and CF are medians drawn from vertices B and C of triangle ABC, intersecting at D, prove triangle BCD equivalent to quadrilateral $AEDF$.



(area BCD = area BCF − area BDF.)

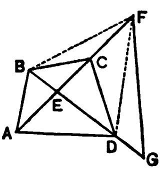
86. If D is the middle point of side BC of triangle ABC, E the middle point of AD, F of BE, and G of CF, prove $\triangle ABC$ equivalent to 8 $\triangle EFG$.

(Draw CE; then, area ABC = 2 area BCE.)

87. If E and F are the middle points of sides AB and CD, respectively, of parallelogram $ABCD$, and AF and CE be drawn intersecting BD in H and L, respectively, and BF and DE intersecting AC in K and G, respectively, prove $GHKL$ a parallelogram equivalent to $\frac{1}{6}ABCD$. (§ 140.)

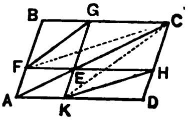
(If AC and BD intersect at M, AM and DE are medians of $\triangle ABD$.)

88. Any quadrilateral $ABCD$ is equivalent to a triangle, two of whose sides are equal to diagonals AC and BD, respectively, and include an angle equal to either of the angles between AC and BD.



(Produce AC to F, making $CF = AE$; and BD to G, making $DG = BE$. To prove quadrilateral $ABCD \approx \triangle EFG$. $\triangle DFG \approx \triangle ABC$.)

89. If through any point E in diagonal AC of parallelogram $ABCD$ parallels to AD and AB be drawn, meeting AB and CD in F and H, respectively, and BC and AD in G and K, respectively, prove triangles EFG and EHK equivalent.



90. If E is the intersection of diagonals AC and BD of a quadrilateral, and triangles ABE and CDE are equivalent, prove sides AD and BC parallel.

(∆s ABD and ACD are equivalent.)

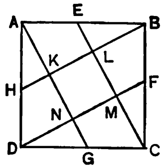
91. Find the area of a trapezoid whose parallel sides are 28 and 36, and non-parallel sides 15 and 17, respectively.

(By drawing through one vertex of the upper base a ∥ to one of the non-parallel sides, one ∠ of the figure may be proved a rt. ∠, by Ex. 63, p. 154.)

92. If similar polygons be described upon the legs of a right triangle as homologous sides, the polygon described upon the hypotenuse is equivalent to the sum of the polygons described upon the legs.

(Find, by § 322, the ratio of the area of the polygon described upon each leg to the area of the polygon described upon the hypotenuse.)

93. If E, F, G, and H are the middle points of sides AB, BC, CD, and DA, respectively, of a square, prove that lines AG, BH, CE, and DF form a square equivalent to $\frac{1}{5}ABCD$.

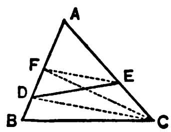


(First prove $\triangle ADG = \triangle ABH$; then, by § 85, 1, $\angle NKL$ may be proved a rt. ∠. By § 131, each side of $KLMN$ may be proved equal to AK. From similar $\triangle$s AHK and ADG, AK may be proved equal to $\frac{AD}{\sqrt{5}}$.)

94. If E is any point in side BC of parallelogram $ABCD$, and DE be drawn meeting AB produced at F, prove triangles ABE and CEF equivalent.

($\triangle ABE + \triangle CDE \approx \triangle CDF$.)

95. If D is a point in side AB of triangle ABC, find a point E in AC such that triangle ADE shall be equivalent to one-half triangle ABC.

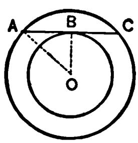


($\triangle DEF \approx \triangle CEF$)

What restriction is there on the position of D?

BOOK V.

96. The area of the ring included between two concentric circles is equal to the area of a circle, whose diameter is that chord of the outer circle which is tangent to the inner.



(To prove area of ring $= \frac{1}{4}\pi \overline{AC}^2$.)

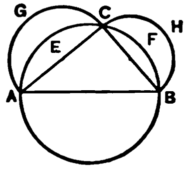
97. An equilateral polygon circumscribed about a circle is regular if the number of its sides is odd. (§ 345.)

(The polygon can be inscribed in a ⊙.)

98. An equiangular polygon inscribed in a circle is regular if the number of its sides is odd. (§ 345.)

(The polygon can be proved equilateral.)

99. If a circle be circumscribed about a right triangle, and on each of its legs as a diameter a semicircle be described exterior to the triangle, the sum of the areas of the crescents thus formed is equal to the area of the triangle. (§ 272.)



(To prove area $AECG$ + area $BFCH$ equal to area ABC.)

100. If the radius of the circle is 1, the side, apothem, and diagonal of a regular inscribed pentagon are, respectively,

$$\tfrac{1}{2}\sqrt{(10-2\sqrt{5})},\ \tfrac{1}{4}(1+\sqrt{5}),\ \text{and}\ \tfrac{1}{2}\sqrt{(10+2\sqrt{5})}.$$

(In Fig. of Prop. IX., the apothem of a regular inscribed pentagon is the distance from O to the foot of a $\perp$ from B to OA, and its side is twice this $\perp$. The diagonal is a leg of a rt. $\triangle$ whose hypotenuse is a diameter, and whose other leg is a side of a regular inscribed decagon.)

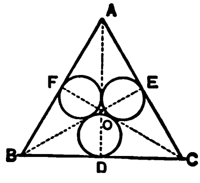
101. The square of the side of a regular inscribed pentagon, minus the square of the side of a regular inscribed decagon, is equal to the square of the radius. (Ex. 100, and § 359.)

102. The sum of the perpendiculars drawn to the sides of a regular polygon from any point within the figure is equal to the apothem multiplied by the number of sides of the polygon.

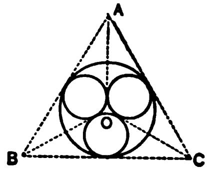
(The ⊥s are the altitudes of $\triangle$s which make up the polygon.)

103. In a given equilateral triangle to inscribe three equal circles, tangent to each other, and each tangent to one, and only one, side of the triangle.

(By § 174, the ⊙s touch the ⊥s at the same points.)



104. In a given circle to inscribe three equal circles, tangent to each other and to the given circle.



ANSWERS

TO

NUMERICAL EXERCISES.

Book I.

4. 24°. **5.** 63° 30′, 26° 30′. **8.** 22° 30′, 157° 30′.
9. 37°. **24.** $A = 112° 30′$, $B = C = 33° 45′$. **88.** 7.

Book II.

12. 28°. **13.** 44° 30′. **14.** 12°. **15.** 54° 30′. **16.** 178°.
17. 112° 30′. **18.** 83°, 89° 30′, 97°, 90° 30′, 74° 30′.
52. $\angle AED = 14° 30′$, $\angle AFB = 10° 30′$.
55. 114° 30′, 89° 30′, 65° 30′, 90° 30′.
67. 97° 30′, 89° 30′, 82° 30′, 90° 30′.

Book III.

1. 112. **2.** 42. **3.** $\frac{25}{27}$. **4.** 63. **5.** BC, $3\frac{1}{5}$, $2\frac{4}{5}$; CA, 4, 3; AB, $4\frac{4}{13}$, $3\frac{9}{13}$. **6.** BC, $11\frac{2}{3}$, $18\frac{2}{3}$; CA, 20, 28; AB, 35, 40.
7. $19\frac{3}{5}$, $25\frac{1}{5}$. **9.** 4 ft. 6 in. **10.** 12. **11.** 15.
12. 37 ft. 1 in. **13.** 47 ft. 6 in. **14.** $\frac{10}{3}\sqrt{3}$.
15. $15\sqrt{2}$ in. **16.** 41. **17.** 58. **18.** 21. **19.** 24.
25. 18. **28.** 48. **29.** 10. **30.** $13\frac{1}{3}$. **31.** $9\sqrt{2}$. **32.** 45.
34. $17\frac{2}{3}$. **37.** 50. **41.** $\sqrt{129}$, $2\sqrt{21}$, $\sqrt{201}$. **42.** $\frac{10}{3}$.
47. 36. **49.** 63. **50.** 4 and 3; $\frac{16}{5}$ and $\frac{9}{5}$. **56.** 24.

38. If $ABCD$ is an inscribed quadrilateral, and sides AD and BC produced meet at P, the tangent at P to the circle circumscribed about triangle ABP is parallel to CD. (§ 196.)

(Prove $\angle$ between the tangent and BP equal to $\angle PCD$.)

39. $ABCD$ is a quadrilateral inscribed in a circle. Another circle is described upon AD as a chord, meeting AB and CD at E and F, respectively. Prove chords BC and EF parallel.

(Prove $\angle ABC = \angle AEF$.)

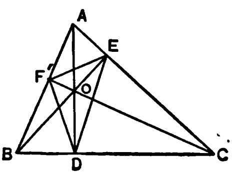
40. If $ABCDEFGH$ is an inscribed octagon, the sum of angles A, C, E, and G is equal to six right angles. (§ 193.)

41. If the number of sides of an inscribed polygon is even, the sum of the alternate angles is equal to as many right angles as the polygon has sides less two.

(Use same method of proof as in Ex. 40.)

42. If a right triangle has for its hypotenuse the side of a square, and lies without the square, the straight line drawn from the centre of the square to the vertex of the right angle bisects the right angle. (§ 200.)

43. The perpendiculars from the vertices of a triangle to the opposite sides are the bisectors of the angles of the triangle formed by joining the feet of the perpendiculars.



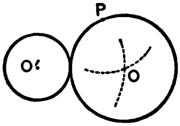
(To prove AD, BE, and CF the bisectors of the $\angle$s of $\triangle DEF$. By § 200, a $\odot$ can be circumscribed about quadrilateral $BDOF$; then $\angle ODF = \angle OBF$; in this way, $\angle ODF = 90° - \angle BAC$.)

Constructions.

44. Given a side, an adjacent angle, and the radius of the circumscribed circle of a triangle, to construct the triangle.

What restriction is there on the values of the given lines?

45. To describe a circle of given radius tangent to a given circle, and passing through a given point without the circle.



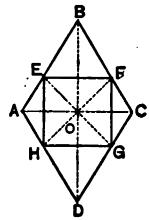
46. To draw between two given intersecting lines a straight line which shall be equal to one given straight line, and parallel to another.

(Draw a $\parallel$ to one of the intersecting lines.)

47. Given an angle of a triangle, the length of its bisector, and the length of the perpendicular from its vertex to the opposite side, to construct the triangle.

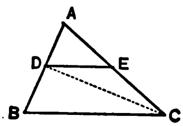
(The side opposite the given $\angle$ is tangent to a $\odot$ drawn with the vertex as a centre, and with the $\perp$ from the vertex to the opposite side as a radius.)

48. Given an angle of a triangle, and the segments of the opposite side made by the perpendicular from its vertex, to construct the triangle. (§ 226.)

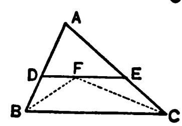


49. To inscribe a square in a given rhombus.

(Bisect the $\angle$s between diagonals AC and BD. To prove $EFGH$ a square, prove $\triangle$s OBE, OBF, ODG, and ODH equal; whence, $OE = OF = OG = OH$.)



50. To draw a parallel to side BC of triangle ABC meeting AB and AC in D and E, respectively, so that DE may equal EC.

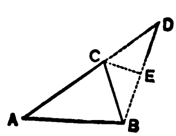


51. To draw a parallel to side BC of triangle ABC, meeting AB and AC in D and E, respectively, so that DE may equal the sum of BD and CE.

52. Given an angle of a triangle, the length of the perpendicular from the vertex of another angle to the opposite side, and the radius of the circumscribed circle, to construct the triangle.

(The centre of the circumscribed $\odot$ is equally distant from the given vertices.)

53. Through a given point without a given circle to draw a secant whose internal and external segments shall be equal. (Ex. 65, p. 103.)



54. Given the base of a triangle, an adjacent angle, and the sum of the other two sides, to construct the triangle.

(Lay off AD equal to the sum of the other two sides.)

59. $3\sqrt[3]{9}$ in., $3\sqrt[3]{18}$ in. **60.** $\dfrac{RT - 2\pi R^3}{2}$. **61.** $\dfrac{4V}{D}$, $\dfrac{8V + \pi D^3}{2D}$. **62.** $\dfrac{\sqrt{9V^2 + 3\pi H^3 V}}{H}$. **63.** $\dfrac{S^2\sqrt{\pi^2 L^4 - S^2}}{3\pi^2 L^3}$.

64. $576\pi\sqrt{2}$. **65.** $\dfrac{S\sqrt{S}}{6\sqrt{\pi}}$. **66.** $\sqrt[3]{36\pi V^2}$.

67. $\dfrac{\pi(a+b)ab}{\sqrt{a^2+b^2}}$, $\dfrac{\pi a^2 b^2}{3\sqrt{a^2+b^2}}$. **68.** 2400π.

69. By triangle, πh^2, $\frac{1}{9}\pi h^3$; by inscribed circle, $\frac{4}{9}\pi h^2$, $\frac{4}{81}\pi h^3$.

70. $\dfrac{4\pi r^2 h^2}{(2r+h)^2}$, $\dfrac{2\pi r^3 h^3}{(2r+h)^3}$. **71.** $2\pi r^2$, $\frac{1}{2}\pi r^3\sqrt{2}$.

72. $2\pi a^2\sqrt{3}$, $\frac{1}{2}\pi a^3$. **73.** 67.3698 + lb. **75.** $2\pi a^2\sqrt{3}$, πa^3.

76. $\frac{640}{3}\pi$, $\frac{400}{3}\pi$. **77.** 2100π. **80.** $\dfrac{2\pi R^2 H}{R+H}$.

81. 1440π. **82.** 487.4716. **83.** $2\pi r^2(1+\sqrt{2})$, $\frac{1}{3}\pi r^3\sqrt{2}$. **86.** 96π.

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